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Generation and Composition of Functions*

A. J. Goldman

(May 18, 1964)

Suppose it is desired to generate some particular function, from a specified set of initial functions, using operations from a specified repertoire. Hypotheses are given which ensure that the process can be so arranged, that the intermediate functions arising at certain stages have no more arguments than does the final function sought.

Operations for producing new functions from old are studied in many branches of mathematics. (In analysis, for example, many of the standard elementary theorems concern the preservation of smoothness properties by such operations.) This topic is especially significant in both the theoretical and the concrete aspects of effective computation, e.g., in recursive-function theory on the one hand and in the programing of digital computers on the other.

Suppose it is desired to generate some particular function, from a specified set of initial functions, using operations from a specified repertoire. It is natural to ask whether the process could be so arranged, that the intermediate functions arising at certain key stages are no more complex (in some appropriate sense) than the final function sought. The present paper deals with some simple topics relating to this question, the "complexity" of a function being measured merely by the number of its arguments.

All *functions* to be considered take values in a common set S, and have finitely many variables which range independently over S. (This last restriction, though awkward in some cases, can sometimes be circumvented by adjoining to S a new element corresponding to "undefined.") A function of k variables will be called a *k-function*. If N is a subset of the natural numbers, then a function which is a k-function for some $k \epsilon N$ will be called an *N-function*.

An *operation* σ is defined to be a mapping whose domain is some subset of the collection of all finite sequences $(f_1, \ldots, f_n)$ of functions, and whose range is a subset of the collection of functions. We call σ an *N-operation* if $\sigma(f_1, \ldots, f_n)$ is defined only if—but not necessarily always if—each f_i is an N-function. For example, if the set S (in which our independent and dependent variables assume their values) happens to be well-ordered, then the operation of *inversion* given by

$$[\sigma_0(f)](x) = \min \{y; f(y) = x\}$$

is a $\{1\}$-operation; $\sigma_0(f)$ is defined if and only if f is a 1-function which assumes all members of S as values.

In any application of an operation σ to a sequence $(f_1, \ldots, f_n)$ in its domain, we call the f_i the *inputs* and $\sigma(f_1, \ldots, f_n)$ the *output*. If Φ is a family of operations, we say that a function f can be *obtained by* Φ from a class C of functions if either $f \epsilon C$ or there exists a finite sequence Σ of applications of operations in Φ, the last of which has f as output, such that the inputs to any of these applications are "available" by virtue either of lying in C or of being the output of some previous application in Σ.

Our final preliminary definition pertains to a family $\Phi_A = \{\sigma_a : a \epsilon A\}$ of operations and a transformation τ which associates to each $\sigma_a \epsilon \Phi_A$ an operation $\tau(\sigma_a)$. The pair (Φ_A, τ) will be called *N-special*, if for each set C of functions closed under all the operations of $\tau(\Phi_A)$, it is true that all N-functions obtainable by Φ_A from C already lie in C. This definition is difficult to motivate here; the reader may wish to look ahead at the definitions preceding Theorem 2, and then at the final paragraph of the paper.

THEOREM 1. *Let the collection* Φ *of operations consist only of* N*-operations and of the operations* σ_a *from some* N*-special pair* (Φ_A, τ). *Then all* N*-functions obtainable by* Φ *from a set* C *of functions are also obtainable if each* $\sigma_a \epsilon \Phi_A$ *is replaced in* Φ *by the corresponding* $\tau(\sigma_a)$.

Proof. Let C_1 consist of all functions obtainable by $(\Phi - \Phi_A) \cup \tau(\Phi_A)$ from C, C_2 consist of all functions obtainable by Φ_A from C_1 (*sic*), and C_3 consist of all functions obtainable by Φ from C. If $C_i(N)$ denotes the class of N-functions in C_i $(i = 1, 2, 3)$, then the statement to be proved is $C_3(N) \subset C_1(N)$. This will be done by showing that

$$C_2(N) \subset C_1, \text{ so that } C_2(N) \subset C_1(N); \tag{1}$$

$$C_3 \subset C_2, \text{ so that } C_3(N) \subset C_2(N). \tag{2}$$

To prove (1), let $f \epsilon C_2$ be an N-function. By construction C_1 is closed under all $\tau(\sigma_a)$, and so the definition of "N-special pair" can be applied to C_1 to assert that

*Helpful comments by R. Kirsch and K. Kloss are gratefully acknowledged.

all N-functions obtainable from C_1 by Φ_A, already lie in C_1. Hence $f \epsilon C_1$ as desired.

To prove (2), observe that C_2 includes C_1 and thus the initial class C, and also is closed under all $\sigma_a \epsilon \Phi_A$; we need only show in addition that it is closed under each $\sigma \epsilon \Phi - \Phi_A$. But if σ (which by hypothesis is an N-operation) is to be applicable to $(f_1, \ldots, f_n)$ where each $f_i \epsilon C_2$, then each f_i must be an N-function and hence $f_i \epsilon C_1$ by (1), implying

$$\sigma(f_1, \ldots, f_n) \epsilon C_1 \subset C_2$$

as desired. This completes the proof of the theorem.

To see how theorem 1 is relevant to the question raised in the second paragraph of the paper, suppose τ can be so chosen relative to Φ_A that the output of any application of any $\tau(\sigma_a)$ is an N-function. If f is any N-function obtainable by Φ from C, then by the theorem f is also obtainable by $(\Phi - \Phi_A) \cup \tau(\Phi_A)$ from C, and in the latter process the intermediate functions resulting from the use of $\tau(\Phi_A)$ are all N-functions; if for example $N = \{1, 2, \ldots, m\}$ where f is an m-function, then these intermediate products are at most m-functions. If in addition the operations in $\Phi - \Phi_A$ produce only N-functions, then *all* the intermediate products are at most m-functions.

Justification of the previous material requires presentation of at least one significant instance to which theorem 1 and the comments of the last paragraph apply. For this purpose we consider the operation σ_c of *composition*, given by

$$[\sigma_c(f_1, \ldots, f_n)](x_1, \ldots, x_k) = f_1(f_2(x_1, \ldots, x_k), \ldots, f_n(x_1, \ldots, x_k))$$

where $n > 1$; here f_1 will be called the *outer* input and the other f_i will be called the *inner* inputs. For any subset N of the natural numbers, we let $\tau_N(\sigma_c)$ be the restriction of σ_c to those sequences $(f_1, \ldots, f_n)$ for which all the inner inputs are N-functions.

[For the following proof, we shall need the observation that the output of an application of composition is a k-function if and only if every inner input is a k-function. The "only if" may seem unduly restrictive; e.g., if

$$f_1(x_1, x_2) = x_1 + x_2, f_2(x_1, x_2) = x_1 x_2, f_3(x_1) = x_1,$$

one would expect to be able to obtain the function $f_4(x_1, x_2) = x_1 x_2 + x_1$ by composition. This is not in general possible with our definition of composition, essentially because there is no mechanism provided for "inflating" f_3 (by adjunction of a dummy variable) to $f_3^*(x_1, x_2) = x_1$. Such inflations *would* be possible if, as is usually assumed in recursive-function theory, the stock of "available" functions includes the generalized identity functions U_i^k defined by

$$U_i^k(x_1, \ldots, x_k) = x_i \qquad (i = 1, 2, \ldots, k).$$

Then we would use two applications of composition,

$$f_4 = \sigma_c(f_1, f_2, f_3^*) = \sigma_c(f_1, f_2, \sigma_c(f_3, U_1^2)),$$

to obtain f_4 from f_1, f_2, and f_3. The functions U_i^k are also necessary if, for example, we wish to obtain $f_5(x_1, x_2, x_3) = x_1 x_2 + x_2 x_3$ from f_1 and f_2:

$$f_5 = \sigma_c(f_1, \sigma_c(f_2, U_1^3, U_2^3), \sigma_c(f_2, U_2^3, U_3^3)).$$

In summary, the composition operation defined above appears "weaker" than usual simply because we do not supplement it by explicitly postulating the availability of the generalized identity functions.]

THEOREM 2. *The pair* $(\{\sigma_c\}, \tau_N)$ *is* N-*special.*

PROOF. Let C be a set of functions closed under $\tau_N(\sigma_c)$, and let f be an N-function obtainable by $\{\sigma_c\}$ from C. We are to prove that f is obtainable by $\{\tau_N(\sigma_c)\}$ from C. It will be convenient to refer to an application of σ_c as an N-composition or an $\overline{N}$-composition, according as all the inner inputs (and hence the output) are N-functions or not.

If $f \epsilon C$, or if there is a sequence of N-compositions leading from C to f, then by the hypothesis on C we have $f \epsilon C$ as desired. (A formal proof would involve induction on the length of the shortest such sequence.) If however every sequence of applications of composition leading from C to f contains at least one $\overline{N}$-composition, then consider such a sequence Σ for which the number of $\overline{N}$-compositions is *minimum*, and let

$$G(y_1, \ldots, y_m) = H(h_1(y_1, \ldots, y_m), \ldots, h_p(y_1, \ldots, y_m)) \quad (3)$$

be the *last* $\overline{N}$-composition in Σ. Thus m is not in N.

By the minimality of Σ, G must be used as an outer or inner input in at least one composition of Σ appearing after (3). It cannot be used as an inner input, since the outputs and hence the inner inputs of all compositions following (3) in Σ are N-functions. Let its *first* use be in

$$F(x_1, \ldots, x_k) = G(g_1(x_1, \ldots, x_k), \ldots, g_m(x_1, \ldots, x_k)). \quad (4)$$

Then $k \epsilon N$, and the availability of the g_j for use just before (4) would not be affected if (3) were deleted from Σ. In addition, the functions H and h_i were available just before (3), and hence would be available just before (4) even if (3) were deleted from Σ.

We show in the next paragraph how to replace (4) in Σ by a sequence of N-compositions, the last of which has F as output. It will follow from the previous remarks that the functions used are in fact available, and would remain so even if (3) were deleted. None of these N-compositions will involve G, so that in the

resulting modification Σ' of Σ (which still leads from C to f) there is one less use of G and the same number of N-compositions. Continuing the process, we arrive at a sequence Σ'' leading from C to f, with the same number of N-compositions as in Σ, and such that (3) appears in Σ'' but is *never* used subsequently. Thus deletion of (3) from Σ'' yields a sequence leading from C to f which contradicts the minimality of Σ.

The sequence of N-compositions which can replace (4) in Σ is given by

$$f_i(x_1, \ldots, x_k) = h_i(g_1(x_1, \ldots, x_k), \ldots, g_m(x_1, \ldots, x_k))$$

for $i = 1, 2, \ldots, p$, followed by

$$F(x_1, \ldots, x_k) = H(f_1(x_1, \ldots, x_k), \ldots, f_p(x_1, \ldots, x_k)).$$

This completes the proof.

Theorems 1 and 2 together immediately imply the following result.

COROLLARY. *If Φ consists of σ_c and N-operations, then all N-functions obtainable by Φ from a set C of functions, are also obtainable if $\tau_N(\sigma_c)$ replaces σ_c in Φ.*

The special case of this corollary in which S consists of the natural numbers, $N = \{1\}$, and Φ consists of σ_c and the inversion operation σ_0 defined earlier, appears in a paper of J. Robinson.[1] The present paper was motivated by a desire to abstract the essentials of this special case. Our proof of Theorem 1 was patterned after the proof given by R. M. Robinson[2] of a precursor of J. Robinson's theorem. Additional interesting applications seem likely to exist, but are apparently difficult to recognize. The subject arose in connection with Davis' characterization[3] of universal Turing machines.

[1] J. Robinson, General recursive functions, Proc. Amer. Math. Soc. **1** (1950).
[2] R. M. Robinson, Primitive recursive functions, Bull. Amer. Math. Soc. **53** (1947).
[3] M. Davis, A note on universal Turing machines, in "Automata Studies," Princeton Annals of Math. Study No. 34.

(Paper 68B3–122)

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Thermodynamics of Perfect Elastic Fluids

B. Bernstein, E. A. Kearsley, and L. J. Zapas

(May 20, 1964)

A simple non-equilibrium thermodynamics is developed and a particular example is studied. The theory is formulated to describe a viscoelastic fluid, capable of finite deformation, which need not be locally in or near a state of thermodynamic equilibrium. This fluid may support shear stresses only when away from local thermodynamic equilibrium. A notion of time-temperature superposition is contained in the formulation of the constitutive equations. Conservation of energy is obeyed and the second law of thermodynamics is satisfied as a consequence of simple requirements on the constitutive relations. In an adiabatic isochoric motion the temperature increases when work is done on the material and decreases when the material does work. For given volume and temperature, entropy decreases when the material is deformed from equilibrium. It is shown in what general way viscosity depends upon temperature. For infinitesimal strain, the special form of the stress-strain relations are derived in order to determine how temperature and time-temperature superposition enter in this case.

1. Introduction

In 1963 we established that a fluid theory gives the correct stress relaxation response for a large variety of materials, some of which, such as vulcanized butyl rubber, are usually treated as solids [1].[1] We started then to investigate whether or not it would be fruitful to regard all matter as fluid. Subsequent experimental work [2] has indicated not only that the fluid point of view can be productive in a practical sense, but that the incompressible elastic fluid theory of our 1963 article is also very promising. It is thus a natural step, at this juncture, to endeavor to extend our theoretical understanding by attempting to create a thermodynamic theory, while deliberately adhering to the fluid point of view and bearing in mind that the incompressible elastic fluid should represent an isothermal limit. We believe that the fluid notion is the key to a simple thermodynamics just as it was the key to the simple isothermal mechanical theory.

We cannot reconcile the notion of a fluid with the notion of equilibrium with shear stresses present. Consequently from this viewpoint, even materials usually considered to be solid, such as vulcanized rubber, cannot be considered in equilibrium while supporting shear. If we are to describe shear, we are therefore left with no alternative other than to write a non-equilibrium thermodynamics.

We offer a caveat. Because the non-equilibrium concept is essential to make the fluid point of view hold together, the reader who reflects on solidlike behavior of our fluid must not forget that we have given up thinking of a sheared state in terms of classical equilibrium thermodynamics. Otherwise he will arrive at paradoxes.

The problem of how to generalize equilibrium thermodynamics to include non-equilibrium relations is resolved by adjoining to the usual state variables a quantity which has dimensions of specific entropy and which depends upon the past history of the material.[2] This quantity is the seat of irreversibility. With it we generate a statement of the second law which is meaningful for non-equilibrium. With it we also arrive at a thermodynamic compressible fluid theory which allows solidlike behavior, and which is consistent with the incompressible elastic fluid theory cited above.

The basic definitions, assumptions, and equations which constitute the thermodynamic theory of perfect elastic fluids are given in section 2. Central to the understanding of these is the idea of local thermodynamic equilibrium or non-equilibrium and the relations which hold in each case. Section 3 is devoted to the manner in which the notion of time-temperature superposition enters the theory. It is suggested that the reader who is anxious to get on to the questions of energy conservation and dissipation may skip or skim through section 3 in his first reading. In section 4 we write down the form which the equation of conservation of energy takes in the context of a perfect elastic fluid. Our statement of fading memory, together with nonnegative heat conductivity, is shown to imply consistency with the second law of thermodynamics in section 5. Also for given specific volume and temperature, the entropy will be less in local non-equilibrium than in local equilibrium. In section 6 we show that positive heat capacity at constant deformation implies that temperature is rising when work is being done on the material and falling when work is being done by the material isochorically and adiabatically. Upper bounds are obtained on the amount of work available by allowing a material to recover either adiabatically

[1] Figures in brackets indicate the literature references on page 112.

or isothermally. In section 7 we derive a relation between temperature and viscosity which has been observed experimentally. And in the final section, section 8, we obtain the stress-strain relations for infinitesimal strain.

2. The Caloric Equation of State

In the theory of inviscid gases, the caloric equation of state relates specific internal energy ϵ to specific volume v and specific entropy s through an expression of the form

$$\epsilon = f(v, s), \tag{2.1}$$

A stipulation of the form of the function f suffices to characterize the gas. The functions relating local pressure p and local absolute temperature T to specific volume and specific entropy are given by

$$p = -f_v(v, s) \equiv -\frac{\partial f}{\partial v} \tag{2.2}$$

$$T = f_s(v, s) \equiv \frac{\partial f}{\partial s}, \tag{2.3}$$

so that one has the familiar relation

$$d\epsilon = Tds - pdv.$$

Although a given volume of gas need not always be in equilibrium because quantities such as temperature or pressure vary from point to point, yet at each point *local thermodynamic equilibrium* is assumed always to hold in the sense that (2.1), (2.2), and (2.3) hold pointwise. When we turn to viscoelastic behavior, we take the point of view that such behavior cannot be explained by local thermodynamic equilibrium, and that a fluid in thermodynamic equilibrium cannot support shear stress. Thus we set ourselves the task of creating a theory which deals with fluids which are not in local thermodynamic equilibrium. We do this with caution, using equilibrium thermodynamic theory (theory of gases) as a basic guide, and departing conceptually from this theory in as little measure as we feel will allow us to explain observed viscoelastic phenomena. The theoretical material thus described will be called a *perfect elastic fluid*.

As will be made clear below, we incorporate into our theory the following notions

1° There exists locally a specific internal energy ϵ and a specific entropy s.

2° There exists locally a non-negative quantity Σ having dimensions of entropy per unit mass, which vanishes at thermodynamic equilibrium. Indeed, local equilibrium at a point prevails if and only if $\Sigma = 0$ at that point.

3° In equilibrium a caloric equation of state of the form (2.1) holds. Whether or not local equilibrium prevails, the caloric equation of state reads

$$\epsilon = f(v, s + \Sigma). \tag{2.4}$$

4° There is a local absolute temperature T which satisfies

$$T = f_s(v, s + \Sigma). \tag{2.5}$$

5° The stress is given by a hydrostatic pressure $-p\delta_{ij}$ plus terms that vanish at equilibrium where

$$p = -f_v(v, s + \Sigma). \tag{2.6}$$

In developing our set of notions into a theory, we have the task of relating Σ and stress to other quantities.[3] To this end, we now discuss some kinematics.

We use Cartesian tensor notation, letting the point $\mathbf{x}$ be given by Cartesian coordinates x_i, $i = 1, 2, 3$. For a given motion, the position of a particle $\mathbf{X}$ at time t, $\mathbf{x}(t)$, can be expressed in terms of its position at time τ, $\mathbf{x}(\tau)$, by a function

$$x_i(t) = x_i[\mathbf{x}(\tau), t, \tau], \tag{2.7}$$

from which we may define the relative deformation gradients

$$x_{ik}(t, \tau) \equiv \frac{\partial x_i(t)}{\partial x_k(\tau)}. \tag{2.8}$$

By a configuration, we simply mean an assignment of position x_i to each particle $\mathbf{X}$.

The Cauchy-Green relative strain tensor $C_{ij}(t, \tau)$ is defined by

$$C_{ij}(t, \tau) = x_{ki}(t, \tau)x_{kj}(t, \tau), \tag{2.9}$$

and the Green-St. Venant relative strain tensor is

$$E_{ij}(t, \tau) = \tfrac{1}{2}[C_{ij}(t, \tau) - \delta_{ij}]. \tag{2.10}$$

For the matrix of $E_{kl}(t, \tau)$ and $C_{kl}(t, \tau)$ we write $\mathbf{E}(t, \tau)$ and $\mathbf{C}(t, \tau)$ respectively.

We return to the question of specifying Σ and stress away from local thermodynamic equilibrium, and proceed by assuming the existence of a function

$$S[\mathbf{E}(\xi, \tau), t]$$

which is defined for all $t \geq 0$ and depends on a strain $\mathbf{E}(\xi, \tau)$ only through

$$I_1(\xi, \tau) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

$$I_2(\xi, \tau) = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2$$

$$I_3(\xi, \tau) = \lambda_1^2\lambda_2^2\lambda_3^2$$

[2] In 1950, P. W. Bridgman published a very perceptive article [3] in which he speculated on some of the properties which a non-equilibrium theory of this type would have to exhibit. We have here a concrete example of such a theory which meets Bridgman's requirements.

[3] The incompressible elastic fluid [1] provided a key as to how to write the expression for stress.

where $\lambda_i^2 = \lambda_i^2(\xi, \tau)$, $i = 1, 2, 3$, are the principal values of $\mathbf{C}(\xi, \tau)$.

We write

$$S_{kl}[\mathbf{E}(\xi, \tau), t] \equiv \frac{\partial S}{\partial E_{kl}} \tag{2.11}$$

$$S_t[\mathbf{E}(\xi, \tau), t] \equiv \frac{\partial S}{\partial t}, \tag{2.12}$$

these notations applying no matter what name is given to the arguments of S.

Furthermore, we shall assume the existence of a material property $b_T > 0$, which scales time intervals in a manner to be made more precise in section 3 below. The quantity b_T depends on absolute temperature T and has the value unity at a given reference temperature T_0. We now assert that the value of Σ at present time t depends upon the past history of deformation and temperature through[4]

$$\Sigma(t) = \int_{-\infty}^{t} S\left[\mathbf{E}(t, \tau), \int_{\tau}^{t} b_T(\xi)d\xi\right] b_T(\tau)d\tau, \tag{2.13}$$

where $b_T(\tau)$ means the value of b_T at time τ. Further we specify that stress σ_{ij} at time t satisfies[5]

$$\sigma_{ij} = -p\delta_{ij} \tag{2.14}$$

$$+\rho T\int_{-\infty}^{t} x_{ik}(t, \tau)x_{jl}(t, \tau)S_{kl}\left[\mathbf{E}(t, \tau), \int_{\tau}^{t} b_T(\xi)d\xi\right] b_T(\tau)d\tau,$$

where $\rho = v^{-1}$ is density, S_{kl} is defined by (2.11) and p by (2.6).

(In (2.14), as subsequently, we do not mention explicitly that quantities such as σ_{ij}, ρ, Σ, and T are evaluated at present time t when the context makes this clear.)

We note that $s + \Sigma$ may be eliminated between (2.5) and (2.6) to express p as a function of specific volume and local temperature, say

$$p = g(v, T), \tag{2.15}$$

whence (2.14) becomes

$$\sigma_{ij} = -g(v, T)\delta_{ij} \tag{2.16}$$

$$+\rho T\int_{-\infty}^{t} x_{ik}(t, \tau)x_{jl}(t, \tau)S_{kl}\left[\mathbf{E}(t, \tau), \int_{\tau}^{t} b_T(\xi)d\xi\right] b_T(\tau)d\tau.$$

The expression (2.14) bears a resemblance to the constitutive equation of an incompressible elastic fluid given by Bernstein, Kearsley, and Zapas, which is a corresponding isothermal incompressible theory. Indeed, for constant temperature

$$\int_{\tau}^{t} b_T(\xi)d\xi = (t - \tau)b_T,$$

in which case the stress is given by (2.14) as if each configuration that the material assumed in the past contributed to the stress as the preferred configuration of an elastic material, each contribution being weighted according to the time elapsed, added up, and modified by the addition of an appropriate hydrostatic pressure.

Since the role that S plays is analogous to that of an elastic strain energy, it is reasonable to assume that

$$S[\mathbf{E}(t, \tau), \xi] \geq 0 \tag{2.17}$$

for all $\xi \geq 0$ and all values of $\mathbf{E}(t, \tau)$ for which S is defined, and that

$$S[\mathbf{0}, \xi] = S[\mathbf{E}(t, t), \xi] = 0 \tag{2.18}$$

for all $\xi \geq 0$, where $\mathbf{0}$ is the zero matrix.

From (2.17), it is clear that whenever S vanishes it assumes a minimum value, whence

$$S_{kl}[\mathbf{E}, \xi] = 0 \text{ whenever } S[\mathbf{E}, \xi] = 0. \tag{2.19}$$

In particular, (2.18) and (2.19) yield

$$S_{kl}[\mathbf{0}, \xi] = 0 \tag{2.20}$$

for all $\xi \geq 0$.

Our theory has been set up to be consistent with 1°, 2°, 3°, and 4°. To show that it is also consistent with 5°, we have but to establish that the integral terms in (2.14) vanish when local thermodynamic equilibrium prevails, i.e., when $\Sigma = 0$. But if $\Sigma = 0$, we see from (2.13), (2.17), and $b_T > 0$, that

$$S\left[\mathbf{E}(t, \tau), \int_{\tau}^{t} b_T(\xi)d\xi\right]$$

vanishes for all $\tau \leq t$. Thus, (2.19) assures us that S_{kl} also vanishes for all $\tau \leq t$ and hence, from (2.14), $\sigma_{ij} = -p\delta_{ij}$, which completes our argument that 5° holds.

In addition to (2.17) and (2.18), we assume *fading memory* in the form

$$S_t \leq 0. \tag{2.21}$$

For an interpretation of the motivation for this assumption, let us consider the case in which temperature, and hence b_T, is constant, so that

$$\int_{\tau}^{t} b_T(\xi)d\xi = (t - \tau)b_T$$

is proportional to the time interval between t and τ. Consider, then, a given value of relative strain $\mathbf{E}$, and consider the contribution to the internal energy which arises if the relative strain computed from the configurations at present time t and past time τ takes this given value. Then, in view of (2.4) and (2.13),

[4] If in further discussions it should prove desirable to have a name for Σ, we suggest calling it *entaxy*.

[5] In equation (3.6) of reference [1], the potential U of the isothermal theory of the incompressible fluid replaces ρTS in eq (2.14) of the present article.

(2.21) says that this contribution cannot increase for larger values of the time interval $t-\tau$.

If up to and including present time the material has been motionless, or even just undeformed, i.e., $\mathbf{E}(t,\tau)=\mathbf{0}$ for $\tau \leqslant t$, then (2.13) and (2.18) imply that Σ vanishes and thus local thermodynamic equilibrium prevails. But, furthermore, we can show that *if our material is kept at constant deformation and at constant temperature from some time* t_1 *onward, it will tend toward local thermodynamic equilibrium,* i.e. *If* $\mathbf{E}(t,\tau)=\mathbf{0}$ *and* $T(t)=T(t_1)$ *for all* $t,\tau \geqslant t_1$, *then*

$$\lim_{t\to\infty}\Sigma(t)=0,$$

where $\Sigma(t)$ *decreases monotonically when* $t>t_1$.

Appendix A is devoted to the proof of this statement.

Example: For an ideal gas

$$\epsilon = Dv^{(1-\gamma)}\exp\frac{s}{C_V},$$

where $D>0, \gamma>1$ and $C_V>0$ are constants. Generalizing this equation slightly to allow a different dependence of ϵ on volume, we may consider

$$\epsilon = Y(v)\exp\frac{s}{C_V}, \tag{2.22}$$

where $Y(v)$ is some given function of v. Starting with (2.22) in place of (2.1), we may form an elastic fluid for which (2.4) becomes

$$\epsilon = Y(v)\exp\frac{1}{C_V}(s+\Sigma), \tag{2.23}$$

where Σ is given by an expression of the form (2.13). Then from (2.23) and (2.5) we obtain

$$\epsilon = C_V T. \tag{2.24}$$

Also (2.15) becomes

$$p=-[\ln Y(v)]' C_V T, \tag{2.25}$$

whence (2.16) reads

$$\sigma_{ij}=[\ln Y(v)]' C_V T\delta_{ij} + \rho T\int_{-\infty}^{t} x_{ik}(t,\tau)x_{jl}(t,\tau)S_{kl}\left[\mathbf{E}(t,\tau),\int_{\tau}^{t} b_T(\xi)d\xi\right] b_T(\tau)d\tau \tag{2.26}$$

A perfect elastic fluid for which (2.23) holds may be called an *ideal elastic fluid.*

3. Time-Temperature Superposition

The idea of time-temperature superposition which we introduce is based on the notion of a clock belonging to each particle of the material. Our assumption is that the material reads its own clock, which is temperature sensitive and runs at a rate b_T as fast as a standard or laboratory clock, which is not temperature sensitive.

If at time $t=0$, the temperature sensitive material clock reads θ_0, then at time t it reads $\theta(t)$ given by

$$\theta(t)=\int_0^t b_T(\xi)d\xi+\theta_0. \tag{3.1}$$

At a constant temperature T, the material clock will advance

$$\theta(t)-\theta(\tau)=(t-\tau)b_T \tag{3.2}$$

when the laboratory clock advances $t-\tau$. At reference temperature T_0 the material clock and the laboratory clock run at the same rate, so that $b_T=1$ and $\theta(t)-\theta(\tau)=t-\tau$.

Looking at (2.15) now, we see that p does not depend explicitly on the past history of deformation and temperature, but only on the present value of temperature and specific volume. Using (2.14) we see then, that $[\sigma_{ij}+p\delta_{ij}]/\rho T$ depends entirely on the histories of the temperature and the strain relative to the present configuration. The concept of time-temperature superposition commonly used in linear viscoelasticity [4] can be stated in terms of these ideas as follows:

Suppose that $\mathbf{x}=\mathbf{f}(t,\mathbf{X})$ is a motion occurring at reference temperature T_0 with corresponding specific volume $v(t,\mathbf{X})$ and stress $\sigma_{ij}(t,\mathbf{X})$ (where $\mathbf{X}$ is a particle label). If this motion is replaced by $\mathbf{x}=\mathbf{f}(b_T t,\mathbf{X})$ at constant temperature T, with corresponding specific volume $\hat{v}(t,\mathbf{X})=v(b_T t,\mathbf{X})$ and corresponding density $\hat{\rho}(t,\mathbf{X})=\rho(b_T t,\mathbf{X})$, then the corresponding stress $\hat{\sigma}_{ij}(t,\mathbf{X})$ will satisfy

$$\frac{\hat{\sigma}_{ij}(t,\mathbf{X})+\hat{p}(t,\mathbf{X})\delta_{ij}}{\hat{\rho}(t,\mathbf{X})T}=\frac{\sigma_{ij}(b_T t,\mathbf{X})+p(b_T t,\mathbf{X})\delta_{ij}}{\rho(b_T t,\mathbf{X})T_0} \tag{3.3}$$

where

$$p(t,\mathbf{X})=g[v(t,\mathbf{X}),T_0]$$

and

$$\hat{p}(t,\mathbf{X})=g[\hat{v}(t,\mathbf{X}),T].$$

It is a simple matter to generalize this statement to apply to situations in which the temperature varies with time. We replace each time increment $d\tau$ by $b_T(\tau)d\tau$ and restate the principle as follows: *Let* $\mathbf{x}=\mathbf{f}(t,\mathbf{X})$ *be a given motion occurring at reference temperature* T_0 *and corresponding specific volume* $v(t)$ *with stress* $\sigma_{ij}(t,\mathbf{X})$. *Suppose that* $\hat{\mathbf{x}}=\mathbf{f}(\theta(t),\mathbf{X})$, *with corresponding specific volume* $\hat{v}(t,\mathbf{X})=v(\theta(t),\mathbf{X})$ *and corresponding density* $\hat{\rho}(t,\mathbf{X})=\rho(\theta(t),\mathbf{X})$ *is a motion occurring at (possibly varying) temperature* $T(t)$ *with corresponding* $b_T(t)$, *so that* $\theta(t)$ *is given by (3.1).* Let $\hat{\sigma}_{ij}(t,\mathbf{X})$ *be the stress corresponding to the second motion. Then*

$$\frac{\hat{\sigma}_{ij}(t,\mathbf{X})+\hat{p}(t,\mathbf{X})\delta_{ij}}{\hat{\rho}(t,\mathbf{X})T}=\frac{\sigma_{ij}(\theta(t),\mathbf{X})+p(\theta(t),\mathbf{X})\delta_{ij}}{\rho(\theta(t),\mathbf{X})T_0} \tag{3.4}$$

where

$$p(t, \mathbf{X}) = g(v(t, \mathbf{X}), T_0)$$

and

$$\hat{p}(t, \mathbf{X}) = g(v(t, \mathbf{X}), T).$$

This is our principle of time-temperature superposition. It is, of course, an assumption subject to experimental verification.

To phrase our principle in a less precise but more intuitive way, suppose an observer sees a motion at temperature T_0. Suppose, then, that at temperature $T(t)$ the observer, reading only the temperature sensitive clock, appears to see the same motion again. Then he will appear to see the same history of

$$(\sigma_{ij} + p\delta_{ij})/\rho T.$$

It is now our job to establish that a perfect elastic fluid is consistent with this principle. To this end, consider the motion $\mathbf{x} = \mathbf{f}(t, \mathbf{X})$ at temperature T_0. Then

$$\frac{\sigma_{ij}(t) + p(t)\delta_{ij}}{\rho(t)T_0} = \int_{-\infty}^{t} \pi_{ij}[x_{kl}(t, \tau), t - \tau]\, d\tau \tag{3.5}$$

where

$$\pi_{ij}[x_{kl}(t, \tau), t - \tau] = x_{ik}(t, \tau)x_{jl}(t, \tau)S_{kl}[\mathbf{E}(t, \tau), t - \tau] \tag{3.6}$$

and we have dropped explicit mention of the particle $\mathbf{X}$.

If for the motion $\mathbf{x}(t) = \mathbf{f}(t, \mathbf{X})$ we write for (2.7)

$$\mathbf{x}(t) = \mathbf{g}[\mathbf{x}(\tau), t, \tau],$$

we shall get for the motion $\hat{\mathbf{x}}(t) = \mathbf{x}(\theta(t)) = \mathbf{f}[\theta(t), \mathbf{X}]$

$$\mathbf{x}(\theta(t)) = \mathbf{g}[\mathbf{x}(\theta(\tau)), \theta(t), \theta(\tau)]$$

$$\hat{\mathbf{x}}(t) = \mathbf{g}[\hat{\mathbf{x}}(\tau), \theta(t), \theta(\tau)]$$

whence the relative deformation gradients for the two motions, namely $x_{kl}[t, \tau]$ and $\hat{x}_{kl}[t, \tau]$ are related by

$$\hat{x}_{kl}(t, \tau) = x_{kl}[\theta(t), \theta(\tau)]. \tag{3.7}$$

Now (3.4) and (3.5) yield,

$$\frac{\hat{\sigma}_{ij}(t) + \hat{p}(t)}{\hat{\rho}(t)\hat{T}(t)} = \frac{\sigma_{ij}(\theta(t)) + p(\theta(t))}{\rho(\theta(t))T_0}$$

$$= \int_{-\infty}^{\theta(t)} \pi_{ij}[x_{kl}(\theta(t), \tau), \theta(t) - \tau]\, d\tau. \tag{3.8}$$

Making the change of variable from τ to ξ, in (3.8), where

$$\tau = \theta(\xi), \qquad d\tau = b_T d\xi,$$

yields

$$\frac{\hat{\sigma}_{ij}(t) + \hat{p}(t)}{\hat{\rho}(t)\hat{T}(t)} = \int_{-\infty}^{t} \pi_{ij}[x_{kl}(\theta(t), \theta(\xi)), \theta(t) - \theta(\xi)]\, b_T(\xi)\, d\xi. \tag{3.9}$$

From (3.1)

$$\theta(t) - \theta(\xi) = \int_{\xi}^{t} b_T(\zeta)\, d\zeta. \tag{3.10}$$

Putting (3.7) and (3.10) into (3.9) yields

$$\hat{\sigma}_{ij} = -\hat{p}\delta_{ij} + \hat{\rho}\hat{T} \int_{-\infty}^{t} \pi_{ij}\left[\hat{x}_{kl}(t, \xi), \int_{\xi}^{t} b_T(\zeta)\, d\zeta\right] b_T(\xi)\, d\xi$$

which, in view of (3.6), is exactly what (2.14) gives for $\hat{\sigma}_{ij}$. Thus the perfect elastic fluid is consistent with our principle of time-temperature superposition.

4. The First Law of Thermodynamics

To write down the first law of thermodynamics it is convenient to distinguish two kinds of derivatives. The *material derivative* of a quantity φ will be written as $\dot{\varphi}$ or $\dfrac{D\varphi}{Dt}$. If we consider φ as a function of time and particle label (Lagrangian coordinates), the material derivative of φ is defined by

$$\dot{\varphi} = \frac{D\varphi}{Dt} \equiv \frac{\partial}{\partial t}\varphi(t, \mathbf{X}).$$

The particle velocity $\mathbf{v}$ is given, for example, by $v_k = \dot{x}_k$. The *spatial derivatives* of a quantity φ will be written $\varphi_{,k}$. If we consider φ as a function of time and position (Eulerian coordinates), the spatial derivatives of φ are defined by

$$\varphi_{,k} = \frac{\partial}{\partial x_k}\varphi(t, \mathbf{x}).$$

Conservation of mass, in the form of the equation of continuity, is written

$$\dot{\rho} + \rho v_{k,k} = 0$$

or, in terms of specific volume

$$\dot{v} = v v_{k,k}$$

whence

$$\rho\dot{v} = v_{k,k} = v_{i,j}\,\delta_{ij}. \tag{4.1}$$

We write the first law of thermodynamics, which expresses conservation of energy, in the form

$$\rho\dot{\epsilon} = \sigma_{ij}\, v_{i,j} - q_{i,i} \tag{4.2}$$

where $\mathbf{q}$ is the *heat flux vector*.

Now material differentiation of (2.4) yields, with the aid of (2.5) and (2.6)

$$\dot{\epsilon} = -p\dot{v} + T\dot{s} + T\dot{\Sigma}. \tag{4.3}$$

Differentiation of (2.13) yields

$$\dot{\Sigma} = S[\mathbf{E}(t,t),0]b_T(t) + \int_{-\infty}^{t} S_{kl}\dot{E}_{kl}b_T(\tau)d\tau + b_T(t)\int_{-\infty}^{t} S_l b_T(\tau)\,d\tau. \tag{4.4}$$

Now from (2.9) and (2.10)

$$\dot{E}_{kl} = \frac{1}{2}\left[\frac{\partial v_i(t)}{\partial x_k(\tau)}\frac{\partial x_i(t)}{\partial x_l(\tau)} + \frac{\partial x_i(t)}{\partial x_k(\tau)}\frac{\partial v_i(t)}{\partial x_l(\tau)}\right] = \frac{1}{2}v_{i,j}\left[\frac{\partial x_j(t)}{\partial x_k(\tau)}\frac{\partial x_i(t)}{\partial x_l(\tau)} + \frac{\partial x_i(t)}{\partial x_k(\tau)}\frac{\partial x_j(t)}{\partial x_l(\tau)}\right], \tag{4.5}$$

where, of course, $v_{i,j} = \partial v_i(t)/\partial x_j(t)$ may be expressed in terms of t and $\mathbf{x}(t)$ alone.

Noting that, by virtue of (2.18), the first term on the right hand side of (4.4) vanishes, we obtain from (4.3), making use of (4.1), (4.4), and (4.5),

$$\rho\dot{\epsilon} = -pv_{i,j}\delta_{ij} + \rho T\dot{s} + \rho T v_{i,j}\int_{-\infty}^{t} x_{ik}(t,\tau)x_{jl}(t,\tau)S_{kl}\left[\mathbf{E}(t,\tau), \int_{\tau}^{t} b_T(\xi)d\xi\right] b_T(\tau)\,d\tau + \rho T b_T(t)\int_{-\infty}^{t} S_l\left[\mathbf{E}(t,\tau), \int_{\tau}^{t} b_T(\xi)d\xi\right] b_T(\tau)d\tau. \tag{4.6}$$

Using (2.14), we get for (4.6)

$$\rho\dot{\epsilon} = \sigma_{ij}v_{i,j} + \rho T\dot{s} + \rho T b_T(t)\int_{-\infty}^{t} S_l b_T(\tau)d\tau. \tag{4.7}$$

Insertion of (4.7) into the first law of thermodynamics (4.2) yields

$$\rho T\dot{s} + q_{i,i} = -\rho T b_T(t)\int_{-\infty}^{t} S_l b_T(\tau)d\tau. \tag{4.8}$$

In the next section we shall show a very important consequence of this equation.

5. The Second Law of Thermodynamics

The law that heat flows against the temperature gradient reads

$$q_i T_{,i} \leqslant 0. \tag{5.1}$$

Coupled with the fact that $b_T > 0$, (2.21) and (4.8) imply

$$\rho T\dot{s} + q_{i,i} \geqslant 0, \tag{5.2}$$

whence, using (5.1)

$$\rho\dot{s} + \frac{q_{i,i}}{T} \geqslant \frac{q_i T_{,i}}{T^2}$$

or

$$\rho\dot{s} + \left(\frac{q_i}{T}\right)_{,i} \geqslant 0,$$

which implies that the Clausius-Duhem inequality holds locally as well as globally [5]. Thus with (5.1), the assumption of fading memory tells us that *a perfect elastic fluid satisfies the second law of thermodynamics in the Clausius-Duhem form.*

Note that we may solve (2.3) for s, obtaining, for equilibrium, an expression of s as a function of v and T, say

$$s = h(v, T).$$

Similarly, whether or not local thermodynamic equilibrium holds, we may solve (2.5) for $s+\Sigma$, obtaining

$$s + \Sigma = h(v, T) \tag{5.3}$$

or

$$s = h(v, T) - \Sigma. \tag{5.4}$$

Recalling 2°, we see immediately from (5.4) the following: *For a given value of* v *and* T, *the local entropy* s *has its maximum value at local thermodynamic equilibrium. Indeed, for a given configuration and* T, *the entropy has its maximum value when local thermodynamic equilibrium prevails.*

Suppose that from some time t_1 onward the material is held at constant deformation, and that the temperature is also maintained constant from t_1 onwards. Then, as can be seen from appendix A, the value of local entropy s will tend towards its equilibrium value $h(v, T)$ in a monotonically increasing manner. Further, (5.3) tells us that $s+\Sigma$ remains constant during this process, any decrease in Σ being exactly equal to an increase in s.

Let us compare the values of local entropy for two cases, one in which local equilibrium prevails ($\Sigma = 0$) and another in which it does not prevail ($\Sigma > 0$). Then if the values of specific volume and temperature are the same in both cases, we shall find that the entropy is less in the latter case than in the former. In particular, at given v and T, the presence of shear stresses indicates that the entropy is strictly less than it would be at equilibrium.

Example: For an ideal elastic fluid (2.23), we get for (5.4)

$$s = C_V \ln\frac{C_V T}{\Upsilon(v)} - \Sigma. \tag{5.5}$$

6. Temperature and Energy of Stretching

Combining (2.4) and (5.3) yields

$$\epsilon = f(v, h(v, T)) \equiv \alpha(v, T), \tag{6.1}$$

where α is appropriately defined. Thus the specific internal energy may be expressed as a function of specific volume and local temperature. This relation holds whether or not local thermodynamic equilibrium prevails.

We define C_Δ by

$$C_\Delta \equiv \frac{\partial \alpha}{\partial T} \tag{6.2}$$

so that from (6.1), (6.2), and (4.2) one has

$$\rho \frac{\partial \alpha}{\partial v} \dot{v} + \rho C_\Delta \dot{T} = \sigma_{ij} v_{i,j} - q_{i,i}, \tag{6.3}$$

which, in view of (4.1), could also be written

$$\rho C_\Delta \dot{T} = \left[\sigma_{ij} - \frac{\partial \alpha}{\partial v} \delta_{ij} \right] v_{i,j} - q_{i,i}. \tag{6.4}$$

Now suppose that heat is flowing but no deformation is occurring so that the symmetric part of $v_{i,j}$ vanishes, i.e., $v_{i,j} + v_{j,i} = 0$. Then (6.4) becomes

$$\rho C_\Delta \dot{T} = -q_{i,i}.$$

Thus C_Δ is the *specific heat capacity at constant deformation* and we feel quite justified in assuming $C_\Delta > 0$.

On the other hand, suppose that the material is adiabatically undergoing isochoric deformation (sometimes called a shearing motion). Then (6.3) becomes

$$\rho C_\Delta \dot{T} = \sigma_{ij} v_{i,j},$$

whence we deduce that *in an adiabatic isochoric motion the temperature is rising or falling depending respectively on whether work is being done on or by the material.*

During an adiabatic deformation the specific work done is equal to the change in specific internal energy as implied by (4.2). Thus an upper bound on the specific amount of work which can be taken out of the material adiabatically is given by ϵ provided that zero is a lower bound on ϵ. Furthermore, when the material is giving up work adiabatically and isochorically, so that $q_{i,i} = 0$, $\dot{v} = 0$, $\sigma_{ij} v_{i,j} < 0$, then by (4.3) and (5.2),

$$\dot{\epsilon} \geqslant T\dot{\Sigma}$$

which together with (4.2) yields

$$T\dot{\Sigma} \leqslant v \sigma_{ij} v_{i,j}.$$

Moreover under these circumstances, as we have just argued, $\dot{T} \leqslant 0$, whence

$$\frac{D(T\Sigma)}{Dt} \leqslant T\dot{\Sigma} \leqslant v \sigma_{ij} v_{i,j} < 0.$$

From this and $T\Sigma \geqslant 0$ it follows that $T\Sigma$ forms an upper bound on the amount of specific work that may be extracted from the material in recovering adiabatically and isochorically.

The specific Helmholtz free energy, F, may be defined by

$$F \equiv \epsilon - Ts. \tag{6.5}$$

From (6.1), (5.4), and (6.5), we get

$$F = \Phi(v, T) + T\Sigma \tag{6.6}$$

where

$$\Phi(v, T) \equiv \alpha(v, T) - Th(v, T).$$

It is clear from (6.6) that for given temperature and specific volume, or alternatively for given temperature and configuration, F takes its minimum when local thermodynamic equilibrium prevails. Also, from appendix A, we see that for constant configuration and temperature, F will tend monotonically to its equilibrium value.

From (6.5), (4.2), and (5.2)

$$\rho \dot{F} = \rho \dot{\epsilon} - \rho T \dot{s} - \rho s \dot{T} = \sigma_{ij} v_{i,j} - \rho s \dot{T} - (\rho T \dot{s} + q_{i,i}) \leqslant \sigma_{ij} v_{i,j} - \rho s \dot{T}. \tag{6.7}$$

Thus, if $\Phi(v, T)$ has a lower bound for each T when $p \geqslant 0$, then the difference between F and this lower bound constitutes an upper bound on the specific amount of work which may be recovered isothermally from the material. In any case, (6.6) and (6.7) tell us that $T\Sigma$ gives an upper bound on the specific amount of work that may be recovered isochorically and isothermally from the material.

If the material is deformed from equilibrium and then allowed to come back to its initial temperature and volume while still in shear, (6.1) tells us that there is no net change in internal energy, and thus, from (4.2), the work done on the material has all been converted into heat, which has been given off to the surroundings. If work is now extracted from the material, and then the material is allowed to return to its original volume and temperature, we similarly see that the material has effectively removed heat from its surroundings and converted it into work. When stresses relax to zero isothermally with no motion of the material, no energy changes or conversions take place. Rather, the material loses its potential to convert heat into work while in contact with isothermal surroundings. The loss of this ability can leave a net increase in the amount of heat that has flowed into the surroundings, giving rise to viscous heat production. Thus, in this theory, elastic behavior and viscous heat production are intimately related in contrast to their frequent introduction as separate phenomena [6].

Example: For an ideal elastic fluid, (2.23), we have already obtained the special form of (6.1) namely

$$\epsilon = C_V T. \tag{6.8}$$

Thus for this fluid

$$C_\Delta = C_V \tag{6.9}$$

and we see that the heat capacity at constant deformation is constant. Indeed, it is a simple exercise to establish that the most general perfect elastic fluid having constant C_Δ is

$$\epsilon = \Upsilon(v) \exp \frac{1}{C_V}[s+\Sigma] + \psi(v) \tag{6.10}$$

where $\Upsilon > 0$ and ψ are functions of v and $C_V > 0$ is a constant. For (6.10), we also have (6.9). For (2.23) we get from (2.24) and (4.2)

$$\rho C_V \dot{T} = \sigma_{ij} v_{i,j} - q_{i,i},$$

whence we see that the temperature will be rising or falling when adiabatic work is done on or by the material respectively. And this will hold whether or not volume changes are occurring.

For (2.23), (6.6) becomes

$$F = C_V T\left[1 - \ln \frac{C_V T}{\Upsilon(v)}\right] + T\Sigma$$

$$\Phi(v, T) = C_V T\left[1 - \ln \frac{C_V T}{\Upsilon(v)}\right]$$

and thus, in light of (2.25), $\Phi(v, T)$ has a lower bound for each given T and all $p \geq 0$ if and only if $\Upsilon(v)$ is bounded away from zero for all values of v for which $[\ln\ \Upsilon(v)]' \leq 0$. In particular, $\Upsilon(v) = [a(v-v_1)^2 + b]^{-1}$, where a and b and v_1 are positive constants, is so bounded. On the other hand, for an ideal gas

$$\Upsilon(v) = Dv^{1-\gamma},\ D > 0,\ \gamma > 1,$$

we have $\Upsilon(v) \to 0$ as $v \to \infty$, whence $\Phi(v, T)$ has no lower bound for any T.

7. Viscosity-Temperature Relations

The motion

$$\begin{aligned} x_1(t) &= x_1(\tau) + (t-\tau)\dot{\gamma} x_2(\tau) \\ x_2(t) &= x_2(\tau) \\ x_3(t) &= x_3(\tau) \end{aligned} \tag{7.1}$$

with $\dot{\gamma}$ constant, represents a simple shear. The relative deformation matrix is

$$\| x_{ik}(t, \tau) \| = \begin{Vmatrix} 1 & (t-\tau)\dot{\gamma} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}$$

and

$$\mathbf{E}(t, \tau) = \frac{1}{2}\begin{Vmatrix} 0 & (t-\tau)\dot{\gamma} & 0 \\ (t-\tau)\dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}$$

Thus we have, for a simple shear (7.1),

$$x_{ik}(t, \tau) x_{jl}(t, \tau) S_{kl}\left[\mathbf{E}(t, \tau), \int_\tau^t b_T(\xi) d\xi\right] = \varphi_{ij}\left[(t-\tau)\dot{\gamma}, \int_\tau^t b_T(\xi) d\xi\right] \tag{7.2}$$

for appropriately defined functions φ_{ij}, $i, j = 1, 2, 3$. Thus from (7.2) and (2.14), under isothermal conditions,

$$\sigma_{12} = \rho T \int_{-\infty}^t \varphi_{12}[(t-\tau)\dot{\gamma}, (t-\tau)b_T] b_T d\tau. \tag{7.3}$$

Making the change of variable $\xi = (t-\tau)b_T$ in (7.3) yields

$$\sigma_{12} = \rho T \int_0^\infty \varphi_{12}[a_T \dot{\gamma}\xi, \xi] d\xi \equiv \rho T H(a_T \dot{\gamma}) \tag{7.4}$$

where $a_T = 1/b_T$ and H is the function defined by (7.4). The viscosity η is defined by

$$\eta = \frac{\sigma_{12}}{\dot{\gamma}}$$

whence, from (7.4)

$$\eta = \rho T \frac{H(a_T \dot{\gamma})}{\dot{\gamma}}. \tag{7.5}$$

The zero shear viscosity η_0 is defined by

$$\eta_0 = \lim_{\dot{\gamma} \to 0} \eta$$

whence, from (7.5)

$$\eta_0 = \rho T a_T \lim_{a_T \dot{\gamma} \to 0} \frac{H(a_T \dot{\gamma})}{a_T \dot{\gamma}} = \rho T a_T H'(0). \tag{7.6}$$

Thus, from (7.5) and (7.6)

$$\frac{\eta}{\eta_0} = \frac{H(a_T \dot{\gamma})}{a_T \dot{\gamma} H'(0)}, \tag{7.7}$$

i.e., η/η_0 *is a function of* $a_T\dot{\gamma}$ *alone*. Now from (7.6)

$$a_T = \frac{\eta_0}{\rho T H'(0)}$$

whence, eliminating a_T in (7.7) yields

$$\frac{\eta}{\eta_0} = \frac{H(\eta_0\dot{\gamma}/\rho T H'(0))}{\eta_0\dot{\gamma}/\rho T} = L\left(\frac{\eta_0\dot{\gamma}}{\rho T}\right) \tag{7.8}$$

where L is an appropriately defined function.

Equation (7.8) is in agreement with experimental data reported by F. Bueche [7].

8. Infinitesimal Strain

Let us expand S in powers of

$$\begin{aligned} J_1 &= tr\mathbf{E}(t,\tau) = E_{kk}(t,\tau) = \tfrac{1}{2}(I_1 - 3) \\ J_2 &= tr\mathbf{E}^2(t,\tau) = E_{kl}(t,\tau)E_{lk}(t,\tau) \\ &= \tfrac{1}{4}(I_1-3)^2 + (I_1-3) - \tfrac{1}{2}(I_2-3) \\ J_3 &= \det\mathbf{E}(t,\tau) = \tfrac{1}{8}[I_3 + I_1 - I_2 - 1]. \end{aligned} \tag{8.1}$$

Because $S[\mathbf{0}, t] = S_{kl}[\mathbf{0}, t] = 0$, we get, to second order in $E_{kl}(t,\tau)$,

$$\begin{aligned} &S\left[\mathbf{E}(t,\tau), \int_\tau^t b_T(\xi)d\xi\right] \\ &\quad = \frac{1}{2}\varphi\left(\int_\tau^t b_T(\xi)d\xi\right)J_1^2 + \frac{1}{2}\psi\left(\int_\tau^t b_T(\xi)d\xi\right)J_2, \end{aligned} \tag{8.2}$$

where φ and ψ are appropriate functions. Thus

$$S_{kl} = \varphi tr\mathbf{E}\delta_{kl} + \psi E_{kl} \tag{8.3}$$

to first order in E_{kl}.

In appendix B, we show that for infinitesimal strain, with small displacement $\mathbf{u}(t)$ from an undeformed configuration, we may take

$$\begin{aligned} x_{ik}(t,\tau) &= \delta_{ik} + u_{i,k}(t) - u_{i,k}(\tau) \\ E_{ij}(t,\tau) &= e_{ij}(t) - e_{ij}(\tau), \end{aligned} \tag{8.4}$$

where

$$e_{ij} = \tfrac{1}{2}[u_{i,j} + u_{i,j}]. \tag{8.5}$$

Assume now that the undeformed configuration is in equilibrium at temperature T and ambient pressure p_0 with specific volume $v_0 = v(p_0, T)$ and density $\rho_0 = \rho(p_0, T)$. Standard infinitesimal strain analysis yields

$$\begin{aligned} \rho(t) &= \rho_0[1 - e_{kk}(t)] \\ v(t) &= v_0[1 + e_{kk}(t)]. \end{aligned} \tag{8.6}$$

Using (2.15), let

$$\kappa(p_0, T) \equiv g_v(v_0, T)v_0, \tag{8.7}$$

where

$$g_v \equiv \frac{\partial g}{\partial v}.$$

Then, to first order in v

$$g(v, T) = p_0 + \frac{\kappa(p_0, T)}{v_0}(v - v_0) = p_0 + \kappa(p_0, T)e_{kk}. \tag{8.8}$$

Substituting (8.3), (8.4), (8.5), (8.6), and (8.8) into (2.16), and dropping all terms of order greater than one in $u_{i,j}$ gives

$$\begin{aligned} \sigma_{ij} = &-p_0\delta_{ij} - \kappa(p_0, T)\delta_{ij} \\ &+ \delta_{ij}T\int_{-\infty}^t \varphi\left(\int_\tau^t b_T(\xi)d\xi\right)b_T(\tau)d\tau\, tre(t) \\ &+ Te_{ij}(t)\int_{-\infty}^t \psi\left(\int_\tau^t b_T(\xi)d\xi\right)b_T(\tau)d\tau \\ &- T\delta_{ij}\int_{-\infty}^t \varphi\left(\int_\tau^t b_T(\xi)d\xi\right)tre(\tau)b_T(\tau)d\tau \\ &- T\int_{-\infty}^t \psi\left(\int_\tau^t b_T(\xi)d\xi\right)e_{ij}(\tau)\,d\tau, \end{aligned} \tag{8.9}$$

where $tre = e_{kk}$. If we separate the stress and strain into pressure parts, $tr\sigma = \sigma_{kk}$, tre, and shear parts, $\tilde{\sigma}_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}tr\sigma$, $\tilde{e}_{ij} = e_{ij} - \frac{1}{3}\delta_{ij}tre$, we get

$$\begin{aligned} \pi(t) &\equiv tr\sigma(t) + p_0 \\ &= tre(t)\left[-\kappa(p_0, T) - \int_{-\infty}^t K'\left(\int_\tau^t b_T(\xi)d\xi\right)b_T(\tau)d\tau\right] \\ &\quad + T\int_{-\infty}^t K'\left(\int_\tau^t b_T(\xi)d\xi\right)tre(\tau)b_T(\tau)d\tau \end{aligned} \tag{8.10}$$

where

$$K(t) = \int_t^\infty [3\varphi(\tau) + \psi(\tau)]d\tau$$

and

$$\begin{aligned} \tilde{\sigma}_{ij}(t) = &-T\tilde{e}_{ij}\int_{-\infty}^t G'\left(\int_\tau^t b_T(\xi)d\xi\right)b_T(\tau)d\tau \\ &+ T\int_{-\infty}^t G'\left(\int_\tau^t b_T(\xi)d\xi\right)\tilde{e}_{ij}(\tau)b_T(\tau)d\tau, \end{aligned} \tag{8.11}$$

where

$$G(t) = \int_t^\infty \psi(\xi)d\xi.$$

In the first integral of (8.11) we make the change of variable

$$\zeta(t, \tau)=\int_\tau^t b_T(\xi)d\xi$$

to get

$$\tilde{\sigma}_{ij}=TG(0)\tilde{e}_{ij}(t)+T\int_{-\infty}^t G'\left(\int_\tau^t b_T(\xi)d\xi\right)\tilde{e}_{ij}(\tau)b_T(\tau)d\tau. \tag{8.12}$$

In changing variables, we have used the assumption that $b_T(\zeta)$ is bounded away from zero so that $\zeta(t, -\infty)$ is infinite. Then (8.12) follows upon observing that $G(\infty)$ is zero. Similarly we get from (8.10)

$$\pi=[-\kappa(p_0, T)+TK(0)]tre(t)+T\int_{-\infty}^t K'\left(\int_\tau^t b_T(\xi)d\xi\right)tre(\tau)b_T(\tau)d\tau. \tag{8.13}$$

In particular, for isothermal stress relaxation, in which $e_{ij}(t)=0$ for $t<0$, $e_{ij}(t)=$ const. for $t>0$, (8.12) and (8.13) yield

$$\tilde{\sigma}_{ij}=TG(b_Tt)\tilde{e}_{ij}$$

$$\pi=[-\kappa(p_0, T)+TK(b_Tt)]tre.$$

Example: For an ideal elastic fluid (2.25) gives us

$$p_0=-[\ln \Upsilon(v_0)]'C_VT \tag{8.14}$$

whence v_0 is a function of p_0/C_VT. Moreover, from (8.7)

$$\kappa(p_0, T)=-[\ln \Upsilon(v_0)]''C_VTv_0. \tag{8.15}$$

Thus we may solve (8.14) for v_0 and insert the result into (8.15) to obtain an expression for $\kappa(p_0, T)$ of the form

$$\kappa(p_0, T)=-\beta\left(\frac{p_0}{C_VT}\right)C_VT,$$

where β is an appropriately defined function.

From this we see that the infinitesimal stress-strain relations for an ideal elastic fluid are given when the three functions G, K, β, and the constant C_V are specified. The relation for the pressure component specializes for an ideal elastic fluid from (8.13) to

$$\pi=\left[\beta\left(\frac{p_0}{C_VT}\right)C_VT+TK(0)\right]tre(t)+T\int_{-\infty}^t K'\left(\int_\tau^t b_T(\xi)d\xi\right)tre(\tau)b_T(\tau)d\tau,$$

but the relation for the shear components does not simplify from (8.12).

9. Discussion

In our presentation of the theory of perfect elastic fluids, we have concentrated on the mathematical consistency of the formulation, and have tried to keep our theory as simple as possible. Without essential changes in the theory, one could set Σ equal to a sum of terms, each similar to the right hand side of (2.13), but each involving a different b_T, or one could let b_T depend upon specific volume v as well as temperature. Numerous other modifications are also possible. We feel that further investigation is needed before deciding whether such modifications are necessary to make our theory describe real materials.

At present it appears to us that our simple formulation can describe elastomers in what rheologists call the rubbery and flow regions. The applicability of the present formulation to the glassy region cannot be decided without further investigation.

Finally we should emphasize that a principal feature of this paper is its new approach to non-equilibrium thermodynamics. It is *not* a theory of perturbation about equilibrium. Nor need there exist an equilibrium path between two given equilibrium states. We feel that the essence of this approach should be applicable to any thermodynamic process.

10. References

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Appendix A

We are to show here that *if* $\mathbf{E}(t, \tau)=0$ *and* $T(t)=T(t_1)$ *for all* t, $\tau \geq t_1$, *then* $\lim_{t\to\infty}\Sigma(t)=0$ *where* $\Sigma(t)$ *decreases monotonically.*

Since $\mathbf{E}(t, \tau)=0$ for $t>t_1$, monotonicity follows from (4.4), (2.18), and (2.21). We have yet to show that $\lim_{t\to\infty}\Sigma=0$.

Let δ be any positive number. For $t \geq t_1$ we have

$$0 \leq \Sigma=\int_{-\infty}^{t_1} S\left[\mathbf{E}(t_1, \tau), \int_\tau^t b_T(\xi)d\xi\right]b_T(\tau)d\tau. \tag{A.1}$$

Because of convergence of the improper integral (A.1) for $t=t_1$, there exists a $t_2 < t_1$ such that

$$\int_{-\infty}^{t_2} S\left[\mathbf{E}(t_1, \tau), \int_{\tau}^{t} b_T(\xi)d\xi\right] b_T(\tau)d\tau < \frac{\delta}{2} \qquad (A.2)$$

and because of (2.21) this must also hold for $t \geq t_1$.

Digress now to consider a motion involving only a single step deformation at $t=0$, such that $T=T_0$. For such a motion Σ becomes at zero time just after the step

$$\int_{-\infty}^{0} S[\mathbf{E}(0, \tau), t-\tau]d\tau$$

where $\mathbf{E}=\mathbf{E}(0, \tau)$ is constant. In order for this integral to converge with $S_t \leq 0$, we must have

$$\lim_{\xi \to \infty} S[\mathbf{E}, \xi] = 0$$

for each value of $\mathbf{E}$, and also this convergence must be monotonic. Returning now to (A.1) we thus see that for each fixed τ, $t_2 \leq \tau \leq t_1$,

$$S\left[\mathbf{E}(t_1, \tau), \int_{\tau}^{t} b_T(\xi)d\xi\right]$$

tends monotonically to zero as $t \to \infty$, whence it must do so uniformly in τ, $t_2 \leq \tau \leq t_1$. (Dini's theorem.) Hence for large enough t

$$\int_{t_2}^{t_1} S\left[\mathbf{E}(t_1, \tau), \int_{\tau}^{t} b_T(\xi)d\xi\right] b_T(\tau)d\tau < \frac{\delta}{2}. \qquad (A.3)$$

Adding (A.2) and (A.3) gives, with the use of (A.1), that for large enough t, we have $|\Sigma| < \delta$ whence our result is established.

(Paper 68B3–123)

Appendix B

We discuss here in a non-rigorous manner the reduction of finite strain quantities to their infinitesimal strain counterparts. We suppose that a particle is displaced from its initial position X_i at time t by $u_i(t, \mathbf{X})$. Thus the position of the particle $\mathbf{X}$ at time t, namely $x_i(t)$ is given by

$$x_i(t) = X_i + u_i(t, \mathbf{X}) \qquad (B.1)$$

whence

$$x_i(t) = x_i(\tau) + u_i(t, \mathbf{X}) - u_i(\tau, \mathbf{X}).$$

Thus

$$x_{ik}(t, \tau) = \delta_{ik} + \frac{\partial X_l}{\partial x_k(\tau)}\left[\frac{\partial u_i(t, \mathbf{X})}{\partial X_l} - \frac{\partial u_i(\tau, \mathbf{X})}{\partial X_l}\right]$$

$$= \delta_{ik} + \frac{\partial X_l}{\partial x_k(\tau)}\frac{\partial x_m(t)}{\partial X_l}\left[u_{i,m}(t) - u_{i,m}(\tau)\right] \qquad (B.2)$$

where $u_{i,m}(t) = \partial u_i(t, \mathbf{X})/\partial x_m(t)$ and $u_{i,m}(\tau) = \partial u_i(\tau, \mathbf{X}/\partial x_m(t)$.

From (B.1)

$$\frac{\partial X_l}{\partial x_k(\tau)} = \delta_{lk} - \frac{\partial u_l(t, \mathbf{X})}{\partial x_k(\tau)} \qquad (B.3)$$

$$\frac{\partial x_m(t)}{\partial X_l} = \delta_{ml} + \frac{\partial u_m(t, \mathbf{X})}{\partial X_l} \qquad (B.4)$$

Putting (B.3) and (B.4) into (B.2) and retaining only terms up to second order in the derivatives of displacement yields

$$x_{ik}(t, \tau) = \delta_{ik} + u_{i,k}(t) - u_{i,k}(\tau) \qquad (B.5)$$

to first order.

Inserting (B.5) into (2.9–10) and retaining only first order terms in the derivatives of displacement yields

$$E_{ij}(t, \tau) = e_{ij}(t) - e_{ij}(\tau)$$

to first order, where e_{ij} is defined by (8.5).

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Zeros of Polynomials in Several Variables and Fractional Order Differences of Their Coefficients

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A classical result of Eneström (if $c_0, c_1, \ldots, c_n$ $(n \geq 1)$ are not all zero and if they satisfy $c_0 \geq c_1 \ldots \geq c_n \geq 0$, then every zero ζ of $\sum_{k=0}^{n} c_k z^k$ satisfies $|\zeta| \geq 1$) is generalized to polynomials in several variables. A result in the same direction, involving fractional order differences of coefficients, is then established.

1. A Generalization of Eneström's Theorem

We begin with the following

THEOREM 1. *Consider a polynomial* $E(z_1, z_2, \ldots, z_p) \equiv \sum_{h_p=0}^{n_p} \ldots \sum_{h_1=0}^{n_1} c_{h_1 \ldots h_p} z_1^{h_1} \ldots z_p^{h_p}$ $(\not\equiv 0)$ *in the complex variables* $z_1, \ldots, z_p$. *Set* $c_{h_1 \ldots h_p} = 0$ *whenever the* h_ν *are integers but some* h_j *does not satisfy* $0 \leq h_j \leq n_j$.[1] *Suppose*

$$\nabla c_{h_1 \ldots h_p} = \sum_{j=1}^{p} c_{h_1 \ldots h_p} - c_{h_1 \ldots h_{j-1} h_j - 1 h_{j+1} \ldots h_p} = p c_{h_1 \ldots h_p} - \sum_{j=1}^{p} c_{h_1 \ldots h_{j-1} h_j - 1 h_{j+1} \ldots h_p} \leq 0 \quad (1)$$

$(h_\nu = 0, 1, \ldots, n_\nu + 1;\ \nu = 1, 2, \ldots, p;\ (h_1 \ldots h_p) \neq (0, \ldots 0))$. If $E(\zeta_1, \ldots, \zeta_p) = 0$, *then at least one* $|\zeta_j|$ *is* ≥ 1.[2]

Proof. A straightforward computation yields

$$\left(p - \sum_{j=1}^{p} z_j\right) E(z_1, \ldots, z_p) \equiv \sum_{h_p=0}^{n_p+1} \ldots \sum_{h_1=0}^{n_1+1} (p c_{h_1 \ldots h_p} - c_{h_1 - 1 h_2 \ldots h_p} - \ldots$$

$$- c_{h_1 \ldots h_{p-1} h_p - 1}) z_1^{h_1} \ldots z_p^{h_p} \equiv p c_{0 \ldots 0} + \sum_{(h_1, \ldots, h_p) \in \sigma} (\nabla c_{h_1 \ldots h_p}) z_1^{h_1} \ldots z_p^{h_p},$$

where σ is the set of all sequences of integers $(h_1, \ldots, h_p)$ with $0 \leq h_\nu \leq n_\nu + 1$, $\nu = 1, 2, \ldots, p$, $(h_1, \ldots h_p) \neq (0, \ldots 0)$. Setting $z_1 = z_2 = \ldots = z_p = 1$, we have $p c_{0 \ldots 0} = -\sum_{(h_1, \ldots, h_p) \in \sigma} (\nabla c_{h_1 \ldots h_p})$, and thus $\left(p - \sum_{j=1}^{p} z_j\right) E(z_1, \ldots, z_p) \equiv \sum_{(h_1, \ldots, h_p) \in \sigma} (\nabla c_{h_1 \ldots h_p}) z_1^{h_1} \ldots z_p^{h_p} - 1)$, from which we infer that $\nabla c_{h_1 \ldots h_p} \neq 0$ for at least one $(h_1 \ldots h_p) \in \sigma$. If $|z_\nu| < 1$ for $\nu = 1, 2, \ldots, p$ then

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[1] Below, instead of repeating the last sentence, we shall merely place the sign * at the appropriate place.

[2] Thus, if, for example, $p = 2$, the curve $E(x, y) = 0$ in the x, y plane does not intersect the square $-1 < x < 1, -1 < y < 1$.

$$\operatorname{Re}\left[\left(p-\sum_{j=1}^{p} z_j\right)E(z_1, \ldots, z_p)\right]=\sum_{(h_1, \ldots, h_p)\in\sigma}(\nabla c_{h_1 \ldots h_p})\operatorname{Re}(z_1^{h_1} \ldots z_p^{h_p}-1)>0,$$ and so $E(z_1, \ldots, z_p)\neq 0$.

When $p=1$, Theorem 1 reduces to a result of Eneström [1][3] stating that if $c_0, c_1, \ldots, c_n (n\geq 1)$ are not all zero and if they satisfy $c_0\geq c_1 \ldots \geq c_n \geq 0$, then every zero ζ of $\sum_{k=0}^{n} c_k z^k$ satisfies $|\zeta|\geq 1$.

2. Fractional Order Differences

The main purpose of this paper is to generalize a result of Cargo and Shisha [2, Theorem 1 and the end of IV] relating the zeros of a polynomial (in one variable) to fractional order differences of its coefficients.

Consider a multisequence $a_{h_1 \ldots h_p}$ ($p\geq 1$ fixed, $h_\nu=\ldots -2, -1, 0, 1, 2, \ldots$; $\nu=1, 2, \ldots, p$). Let $\mathbf{I}a_{h_1 \ldots h_p}\equiv a_{h_1 \ldots h_p}$, and for $j=1, 2, \ldots, p$, let $\mathbf{E}_j a_{h_1 \ldots h_p}\equiv a_{h_1 \ldots h_{j-1} h_j+1 h_{j+1} \ldots h_p}$. Let α be a complex number. From (1), one is led to the following symbolic equalities

$$\nabla^\alpha=\left(p\mathbf{I}-\sum_{j=1}^{p}\mathbf{E}_j^{-1}\right)^\alpha=p^\alpha\left(\mathbf{I}-p^{-1}\sum_{j=1}^{p}\mathbf{E}_j^{-1}\right)^\alpha=p^\alpha\sum_{m=0}^{\infty}(-1)^m\binom{\alpha}{m}p^{-m}\left(\sum_{j=1}^{p}\mathbf{E}_j^{-1}\right)^m$$

$$=\sum_{m=0}^{\infty}\sum_{\substack{i_k\geq 0\\ \Sigma i_k=m}}(-1)^m m!\binom{\alpha}{m}p^{\alpha-m}[i_1!\ldots i_p!]^{-1}\mathbf{E}_1^{-i_1}\ldots\mathbf{E}_p^{-i_p}.$$

Thus, we define

$$\nabla^\alpha a_{h_1 \ldots h_p}=\sum_{m=0}^{\infty}\sum_{\substack{i_k\geq 0\\ \Sigma i_k=m}}(-1)^m m!\binom{\alpha}{m}p^{\alpha-m}[i_1!\ldots i_p!]^{-1}a_{h_1-i_1 \ldots h_p-i_p}$$

for every $(h_1, \ldots, h_p)$ for which the last infinite sum converges. If $a_{\nu_1 \ldots \nu_p}=0$ whenever some ν_j is <0, and if $h_j\geq 0, j=1, 2, \ldots, p$, then

$$\nabla^\alpha a_{h_1 \ldots h_p}=\sum_{m=0}^{h_1+\ldots+h_p}\sum_{\substack{i_k\geq 0\\ \Sigma i_k=m}}(-1)^m m!\binom{\alpha}{m}p^{\alpha-m}[i_1!\ldots i_p!]^{-1}a_{h_1-i_1 \ldots h_p-i_p}.$$

THEOREM 2. *Consider a polynomial* $E(z_1, \ldots, z_p)\equiv\sum_{h_p=0}^{n_p}\ldots\sum_{h_1=0}^{n_1}c_{h_1 \ldots h_p}z_1^{h_1}\ldots z_p^{h_p}\not\equiv 0)$ *in the complex variables* $z_1, \ldots, z_p$.* *Let* $0<\alpha<1$, *and suppose that* $c_{h_1 \ldots h_p}\geq 0 (h_\nu=0, 1, \ldots, n_\nu; \nu=1, 2, \ldots, p)$, $\nabla^\alpha c_{h_1 \ldots h_p}\leq 0 (h_\nu=0, 1, \ldots n_\nu; \nu=1, 2, \ldots, p; (h_1, \ldots, h_p)\neq(0, \ldots, 0))$. *If* $E(\zeta_1, \ldots, \zeta_p)=0$, *then at least one* $|\zeta_j|$ *is* >1.

Proof. For $\left|\sum_{j=1}^{p} z_j\right|\leq p$ we have [4]

$$\left(1-p^{-1}\sum_{j=1}^{p}z_j\right)^\alpha E(z_1, \ldots, z_p)=\left[\sum_{j=0}^{\infty}\binom{\alpha}{j}(-1)^j p^{-j}\left(\sum_{\nu=1}^{p}z_\nu\right)^j\right]\sum_{j=0}^{\infty}\sum_{\substack{h_k\geq 0\\ \Sigma h_k=j}}c_{h_1 \ldots h_p}z_1^{h_1}\ldots z_p^{h_p}$$

$$=\sum_{j=0}^{\infty}\sum_{m=0}^{j}\left\{\sum_{\substack{i_k\geq 0\\ \Sigma i_k=m}}(-1)^m m!\binom{\alpha}{m}p^{-m}[i_1!\ldots i_p!]^{-1}z_1^{i_1}\ldots z_p^{i_p}\right\}\left\{\sum_{\substack{h_k\geq 0\\ \Sigma h_k=j-m}}c_{h_1 \ldots h_p}z_1^{h_1}\ldots z_p^{h_p}\right\}$$

[3] Figures in brackets indicate the literature references at the end of this paper.

[4] By z^α, for a complex z, we mean the principal value of that power (0 if $z=0$).

$$= \sum_{j=0}^{\infty} \sum_{m=0}^{j} \sum_{\substack{i_k \geq 0 \\ \Sigma i_k = m \\ q_k \geq i_k \\ \Sigma q_k = j}} (-1)^m m! \binom{\alpha}{m} p^{-m} [i_1! \ \ldots \ i_p!]^{-1} c_{q_1 - i_1 \ldots q_p - i_p} z_1^{q_1} \ldots z_p^{q_p}$$

$$= \sum_{j=0}^{\infty} \sum_{m=0}^{j} \sum_{\substack{q_k \geq 0 \\ \Sigma q_k = j}} \sum_{\substack{i_k \geq 0 \\ \Sigma i_k = m}} (-1)^m m! \binom{\alpha}{m} p^{-m} [i_1! \ \ldots \ i_p!]^{-1} c_{q_1 - i_1 \ldots q_p - i_p} z_1^{q_1} \ldots z_p^{q_p}$$

$$= \sum_{j=0}^{\infty} \sum_{\substack{q_k \geq 0 \\ \Sigma q_k = j}} \sum_{m=0}^{\infty} \sum_{\substack{i_k \geq 0 \\ \Sigma i_k = m}} (-1)^m m! \binom{\alpha}{m} p^{-m} [i_1! \ \ldots \ i_p!]^{-1} c_{q_1 - i_1 \ldots q_p - i_p} z_1^{q_1} \ldots z_p^{q_p}$$

$$= \sum_{j=0}^{\infty} \sum_{\substack{q_k \geq 0 \\ \Sigma q_k = j}} p^{-\alpha} (\nabla^{\alpha} c_{q_1 \ldots q_p}) z_1^{q_1} \ldots z_p^{q_p}.$$

Setting $z_1 = \ldots = z_p = 1$, we get $0 = p^{-\alpha} \nabla^{\alpha} c_{0 \ldots 0} + \sum_{j=1}^{\infty} \sum_{\substack{q_k \geq 0 \\ \Sigma q_k = j}} p^{-\alpha} \nabla^{\alpha} c_{q_1 \ldots q_p}$.

Thus, for $|\sum_{j=1}^{p} z_j| \leq p$ we have

$$\left(p - \sum_{j=1}^{p} z_j\right)^{\alpha} E(z_1, \ldots, z_p) = \sum_{j=1}^{\infty} \sum_{\substack{q_k \geq 0 \\ \Sigma q_k = j}} (\nabla^{\alpha} c_{q_1 \ldots q_p}) (z_1^{q_1} \ldots z_p^{q_p} - 1). \tag{2}$$

Let $q_1, \ldots, q_p$ be arbitrary integers ≥ 0 (not all zero). Then $\nabla^{\alpha} c_{q_1 \ldots q_p} \leq 0$. To see this we may assume that some q_j is $> m_j$. But then, noting that $(-1)^m \binom{\alpha}{m} < 0$ for $m = 1, 2, \ldots$, we have

$$\nabla^{\alpha} c_{q_1 \ldots q_p} = \sum_{m=1}^{q_1 + \ldots + q_p} \sum_{\substack{i_k \geq 0 \\ \Sigma i_k = m}} (-1)^m m! \binom{\alpha}{m} p^{\alpha - m} [i_1! \ \ldots \ i_p!]^{-1} c_{q_1 - i_1 \ \ldots \ q_p - i_p} \leq 0.$$

From (2) we obtain, for $\left|\sum_{j=1}^{p} z_j\right| \leq p$,

$$\operatorname{Re}\left\{\left(p - \sum_{j=1}^{p} z_j\right)^{\alpha} E(z_1, \ldots, z_p)\right\} = \sum_{j=1}^{\infty} \sum_{\substack{q_k \geq 0 \\ \Sigma q_k = j}} (\nabla^{\alpha} c_{q_1 \ldots q_p}) \operatorname{Re}(z_1^{q_1} \ldots z_p^{q_p} - 1). \tag{3}$$

Suppose $E(\zeta_1, \ldots, \zeta_p) = 0$, where all $|\zeta_j|$ are ≤ 1. If $q_1, \ldots, q_p$ are non-negative integers (not all zero), then $\operatorname{Re}(\zeta_1^{q_1} \ldots \zeta_p^{q_p} - 1) \leq 0$, and equality must hold if $\nabla^{\alpha} c_{q_1 \ldots q_p} < 0$. Let $j (1 \leq j \leq p)$ be an integer. Since some $c_{h_1 \ldots h_p}$ must be positive, we have

$$\nabla^{\alpha} c_{n_1 \ldots n_{j-1} n_j + 1 n_{j+1} \ldots n_p}$$

$$= \sum_{m=1}^{n_1 + \ldots + n_p + 1} \sum_{\substack{i_k \geq 0 \\ \Sigma i_k = m}} (-1)^m m! \binom{\alpha}{m} p^{\alpha - m} [i_1! \ \ldots \ i_p!]^{-1} c_{n_1 - i_1 \ldots n_{j-1} - i_{j-1} n_j + 1 - i_j n_{j+1} - i_{j+1} \ \ldots \ n_p - i_p} < 0,$$

and similarly $\nabla^{\alpha} c_{n_1 \ \ldots \ n_{j-1} \ n_j + 2 n_{j+1} \ \ldots \ n_p} < 0$. Thus

$\mathrm{Re}\,(\zeta_1^{n_1} \cdots \zeta_{j-1}^{n_{j-1}}\zeta_j^{n_j+1}\zeta_{j+1}^{n_{j+1}} \cdots \zeta_p^{n_p}-1)=\mathrm{Re}\,(\zeta_1^{n_1} \cdots \zeta_{j-1}^{n_{j-1}}\zeta_j^{n_j+2}\zeta_{j+1}^{n_{j+1}} \cdots \zeta_p^{n_p}-1)=0$, and so

$$\zeta_1^{n_1} \cdots \zeta_{j-1}^{n_{j-1}}\zeta_j^{n_j+1}\zeta_{j+1}^{n_{j+1}} \cdots \zeta_p^{n_p}=1=\zeta_1^{n_1} \cdots \zeta_{j-1}^{n_{j-1}}\zeta_j^{n_j+2}\zeta_{j+1}^{n_{j+1}} \cdots \zeta_p^{n_p},$$

which implies that $\zeta_j=1$. Thus $\zeta_1=\zeta_2=\cdots=\zeta_p=1$, and so $0=E(1,1,\ldots,1)$ $=\sum_{h_p=0}^{n_p} \cdots \sum_{h_1=0}^{n_1} c_{h_1 \ldots h_p}>0$, which proves the Theorem.

3. Remarks

We consider a polynomial $E(z_1, \ldots, z_p) \equiv \sum_{h_p=0}^{n_p} \cdots \sum_{h_1=0}^{n_1} \alpha_{h_1 \ldots h_p} z_1^{h_1} \cdots z_p^{h_p}$ $(\not\equiv 0)$ in the complex variables $z_1, \ldots, z_p$.

A. Suppose the $\alpha_{h_1 \ldots h_p}$ are complex. Let $r_1, \ldots, r_p$ be complex, non-zero numbers, and set $c_{h_1 \ldots h_p}=\alpha_{h_1 \ldots h_p} r_1^{h_1} \cdots r_p^{h_p} (h_\nu=0, 1, \ldots, n_\nu;\ \nu=1, 2, \ldots, p)$.* Assume $\nabla c_{h_1 \ldots h_p} \leq 0$ $(h_\nu=0, 1, \ldots, n_\nu+1;\ \nu=1, 2, \ldots, p;\ (h_1, \ldots, h_p) \neq (0, \ldots 0))$. Suppose $E(\zeta_1, \ldots, \zeta_p)=0$. Then at least one $|\zeta_j|$ is $\geq |r_j|$. Indeed,

$$E(z_1, \ldots, z_p) \equiv \sum_{h_p=0}^{n_p} \cdots \sum_{h_1=0}^{n_1} c_{h_1 \ldots h_p} \left(\frac{z_1}{r_1}\right)^{h_1} \cdots \left(\frac{z_p}{r_p}\right)^{h_p},$$

and therefore by Theorem 1, at least one $|\zeta_j/r_j|$ is ≥ 1.

As a simple example, take $p=1$, $n_1=n(\geq 1)$, and assume the α_ν are alternating in sign: $\alpha_\nu=(-1)^\nu|\alpha_\nu|(\nu=0, 1, \ldots, n)$ and $|\alpha_0| \geq |\alpha_1| \cdots \geq |\alpha_n|$.

By taking $r_1=-1$, we obtain that every zero of the polynomial $\sum_{\nu=0}^{n} \alpha_\nu z^\nu$ has a modulus ≥ 1.

B. Suppose that $E(z_1, \ldots, z_p)$ is not a constant and that the $\alpha_{h_1 \ldots h_p}$ are ≥ 0. Let

$$r=\min\left[\left(\sum_{\substack{j=1 \\ h_j \geq 1}}^{p} \alpha_{h_1 \ldots h_{j-1} h_j-1 h_{j+1} \ldots h_p}\right) \Big/ p\alpha_{h_1 \ldots h_p}\right]$$

where the minimum is taken over all sequences $(h_1, \ldots, h_p)$ with $0 \leq h_\nu \leq n_\nu(\nu=1, 2, \ldots, p)$, $(h_1, \ldots, h_p) \neq (0, \ldots, 0)$, $\alpha_{h_1 \ldots h_p} \neq 0$. Suppose $E(\zeta_1, \ldots, \zeta_p)=0$. Then at least one $|\zeta_j|$ is $\geq r$. Indeed, we may suppose $r>0$. Let $c_{h_1 \ldots h_p}=\alpha_{h_1 \ldots h_p} r^{h_1+\cdots+h_p} (h_\nu=0, 1, \ldots, n_\nu;\ \nu=1, 2, \ldots, p)$.* Let $0 \leq h_\nu \leq n_\nu+1$, $\nu=1, 2, \ldots, p$; $(h_1, \ldots h_p) \neq (0, \ldots, 0)$. We shall show that $\nabla c_{h_1 \ldots h_p} \leq 0$. We may clearly assume that all h_ν are $\leq n_\nu$ and that $\alpha_{h_1 \ldots h_p} \neq 0$.

Then $\nabla c_{h_1 \ldots h_p}=r^{h_1+\cdots+h_p-1}\left[rp\alpha_{h_1 \ldots h_p}-\sum_{\substack{j=1 \\ h_j \geq 1}}^{p} \alpha_{h_1 \ldots h_{j-1} h_j-1 h_{j+1} \ldots h_p}\right] \leq 0$. By A, at least one $|\zeta_j|$ is $\geq r$.

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Theory of Radiation From Sources Immersed in Anisotropic Media

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The electromagnetic fields produced by an electric dipole immersed in an anisotropic medium are considered. Various approaches to the problem are outlined with special reference to a cold plasma. An attempt is made to show the close relationship between previously published work on this subject. It is shown that information on the radiation field in an anisotropic media may be obtained directly from the shape of the refractive index surface.

1. Introduction

There is now a vast amount of literature on the subject of electromagnetic waves in anisotropic media. However, until quite recently, only plane wave solutions have been considered in any detail. In most applications to the ionosphere, this approach has been justified since the transmitting and receiving antennas were ground based. In recent years, the situation is gradually changing since, increasingly often, the source is placed within the anisotropic medium. The best example is when a transmitter is located in a satellite which has an antenna attached to it in some manner.

It is the purpose of this paper to review some of the recent theoretical investigations on the subject of radiation from sources in anisotropic media. It is assumed throughout that nonlinear phenomena may be ignored. Thus, the medium is characterized by a tensor dielectric constant and a tensor magnetic permeability whose elements are independent of the field.

In the first part of this paper, the Green's function for a generally anisotropic (i.e., gyrotropic) media is considered. The results are then specialized to a medium whose magnetic properties are isotropic. This important special case corresponds to a cold plasma which may be described as a gyroelectric medium. Using this type of medium as an example, some alternative approaches to solving for the fields are described. The extensions required to treat stratified anisotropic media are also described briefly. Finally, the special problems associated with asymptotic evaluation of the integrals are considered.

2. Green's Functions for Generally Anisotropic Medium

The present development of the Green's function is closely related to that of Bunkin [1957], Kogelnik [1960], Meecham [1961], Kuehl [1962], and Ament [1963]. In particular, following Ament [1963], we assume a Cartesian coordinate system (x_1, x_2, x_3) in which E_j, H_j, and J_j represent the component of the electric field, magnetic field and source current density, parallel to the x_j axis. As usual, all field quantities vary according to $\exp(i\omega t)$.

The medium itself is considered to be homogeneous and generally anisotropic. Thus, the dielectric constant and magnetic permeability are denoted by tensors (ϵ) and (μ), independent of position. For this situation, Maxwell's equations for the jth component are written

$$(\vec{\nabla} \times \vec{E})_j + i\omega\mu_{jn}H_n = 0, \tag{1}$$

$$(\vec{\nabla}\times\vec{H})_j - i\omega\epsilon_{jn}E_n = J_j, \tag{2}$$

where $\mu_{jn}=(\mu)$ and $\epsilon_{jn}=(\epsilon)$ are three-by-three matrices or dyads with constant elements.

Formally, the field $E_i(\vec{x})$ at the vector position $\vec{x}$ may be obtained via a Green's function $\tilde{G}_{ij}(\vec{x}, \vec{x'})$ from the current density $J_j(\vec{x'})$ at vector position $\vec{x'}$. One should note that the components of $\vec{x}$ are x_1, x_2, x_3 and those of $\vec{x'}$ are x_1', x_2', x_3'. The Green's function $\tilde{G}_{ij}$ is a tensor and is a function of both $\vec{x}$ and $\vec{x'}$. It is defined by the relation

$$E_i(\vec{x}) = \int\int\int \tilde{G}_{ij}(\vec{x}, \vec{x'})\, J_j(\vec{x'})\, dx_1' dx_2' dx_3', \tag{3}$$

where the integration is over any volume which encloses the source current density. A more compact description of (3) is written

$$E_i(\vec{x}) = \int G_{ij}(\vec{x}, \vec{x'}) J_j(\vec{x'}) d^3x'.$$

To obtain an expression for the Green's function, it is convenient to write Maxwell's equations in symbolic form. For example, if ∂_i symbolizes the derivative $\partial/\partial x_i$, it is evident that

$$\vec{\nabla}\times\vec{E} = -i(\mu)\omega\vec{H} \tag{4}$$

is equivalent to

$$\begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = -i\omega \begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{pmatrix} \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix}, \tag{5}$$

or, simply

$$\tilde{\partial}\cdot\vec{E} = -i\omega\tilde{\mu}\cdot\vec{H}. \tag{6}$$

Similarly, the other of Maxwell's equations,

$$\vec{\nabla}\times\vec{H} = i(\epsilon)\omega\vec{E} + \vec{J} \tag{7}$$

becomes

$$\tilde{\partial}\cdot\vec{H} = i\omega\tilde{\epsilon}\cdot\vec{E} + \vec{J}. \tag{8}$$

The vector $\vec{H}$ may now be eliminated from (6) and (8) to obtain

$$\tilde{\partial}\cdot\tilde{\nu}\cdot\tilde{\partial}\cdot\vec{E} - \omega^2\tilde{\epsilon}\cdot\vec{E} = -i\omega\vec{J}, \tag{9}$$

where

$\tilde{\nu} = \dfrac{1}{\tilde{\mu}}$ is the matrix inverse to $\tilde{\mu}$.

To facilitate the solution of (9), it is assumed that both $\vec{E}$ and $\vec{J}$ have three-dimensional Fourier representations. These may be regarded as superpositions of plane waves propagating as exp $[-ik_1x_1-ik_2x_2-ik_3x_3]=\exp[-i\vec{k}\cdot\vec{x}]$ where $\vec{k}$ is the (vector) wave number. Thus,

$$e_n(\vec{k})=(2\pi)^{-3}\int E_n(\vec{x})e^{+i\vec{k}\cdot\vec{x}}d^3x \tag{10}$$

is the transform of $E_n(\vec{x})$ when $n=1$, 2, or 3. The inverse transform is given by

$$E_n(\vec{x})=\int e_n(\vec{k})e^{-i\vec{k}\cdot\vec{x}}d^3k, \tag{11}$$

which is a volume integral over k space. The transform of $J_n(\vec{x})$, which is denoted by $j_n(\vec{k})$, is defined in the same fashion.

The differentiations in (9), indicated by the $\tilde{\partial}$ tensors, may be carried out under the integral signs. Then, on equating coefficients of exp $(-i\vec{k}\cdot\vec{x})$, we obtain

$$\tilde{K}\cdot\tilde{\nu}\cdot\tilde{K}\cdot\vec{e}+\omega^2\tilde{\epsilon}\cdot\vec{e}=i\omega\vec{j}, \tag{12}$$

where $\tilde{K}$ represents the matrix

$$\begin{pmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{pmatrix}.$$

Equation (12) may be solved formally to yield

$$\vec{e}(\vec{k})=i\omega\tilde{M}\cdot\vec{j}(\vec{k}), \tag{13}$$

where

$$\tilde{M}=\frac{1}{\tilde{K}\cdot\tilde{\nu}\cdot\tilde{K}+\omega^2\tilde{\epsilon}}. \tag{14}$$

From (11) we then find that

$$\vec{E}(\vec{x})=i\omega\int e^{-i\vec{k}\cdot\vec{x}}\tilde{M}\cdot\vec{j}(\vec{k})d^3k \tag{15}$$

$$=i\omega\int e^{-i\vec{k}\cdot\vec{x}}\tilde{M}\cdot\int J(\vec{x'})e^{+i\vec{k}\cdot\vec{x'}}(2\pi)^{-3}d^3x'd^3k. \tag{16}$$

This result can be written in the required form

$$\vec{E}(\vec{x})=\int\tilde{G}(\vec{x},\ \vec{x'})\cdot\vec{J}(\vec{x'})d^3x', \tag{17}$$

if

$$\tilde{G}(\vec{x},\vec{x'})=\frac{i\omega}{(2\pi)^3}\int e^{-i\vec{k}\cdot(\vec{x}-\vec{x'})}\tilde{M}d^3k, \tag{18}$$

and provided the order of the x' and the k integration may be interchanged. In matrix notation,

the Green's function is written

$$G_{ij}(\vec{x}, \vec{x'}) = \frac{i\omega}{(2\pi)^3} \int e^{-i\vec{k} \cdot (\vec{x} - \vec{x'})} M_{ij} d^3k, \tag{19}$$

which involves the $\vec{k}$ – dependent matrix M_{ij}.

It may be easily shown that M_{ij} is an even function of $\vec{k}$ in that it has the same value when $\vec{k}$ (or k_1, k_2, k_3) is replaced by $-\vec{k}$ (or by $-k_1$, $-k_2$, $-k_3$). Thus, on noting that M_{ij} is an explicit function of $(\vec{x} - \vec{x'})$ and $\vec{k}$, it is seen that

$$G_{ij}(\vec{x} - \vec{x'}) = G_{ij}(\vec{x'} - \vec{x}). \tag{20}$$

3. Green's Function for Gyroelectric Medium

The development of the Green's function as given above is applicable to homogeneous but generally anisotropic media. The important special case is when the magnetic properties are isotropic. In this instance,

$$(\tilde{\mu}) = \begin{pmatrix} \mu_0 & 0 & 0 \\ 0 & \mu_0 & 0 \\ 0 & 0 & \mu_0 \end{pmatrix}, \tag{21}$$

where μ_0 is the usual magnetic permeability of the media. In most ionospheric-type plasma, it is permissible to regard μ_0 as the permeability of free space. However, this is not an essential restriction on the theory.

A further simplification may be made if the d-c magnetic field is directed along the z axis of the Cartesian coordinate system. The tensor dielectric constant then has the form

$$(\tilde{\epsilon}) = \begin{pmatrix} \epsilon & -ig & 0 \\ ig & \epsilon & 0 \\ 0 & 0 & \hat{\epsilon} \end{pmatrix}, \tag{22}$$

where the (complex) constants ϵ, $\hat{\epsilon}$, and g depend on the properties of the medium.

While the specific form of the Green's function for the gyroelectric case may be obtained as a special case, it is instructive to derive it directly following the method of Kogelnik [1960].

The relevant wave equation, corresponding to (9), is now written in vector-dyadic notation,

$$\left(\vec{\nabla}\,\vec{\nabla} - \Delta\,\tilde{1} - \frac{\omega^2}{c^2}\frac{\tilde{\epsilon}}{\epsilon_0}\right)\vec{E} = -i\mu_0\omega\vec{J}, \tag{23}$$

where $\Delta = \vec{\nabla} \cdot \vec{\nabla}$ is the Laplacian operator, $\tilde{1}$ is the unit matrix or dyad, and $\vec{\nabla}\,\vec{\nabla}$ and $\tilde{\epsilon}$ are also dyads. The elements of $\vec{\nabla}\,\vec{\nabla}$ are the differential operators $\partial^2/\partial x_i \partial x_k$.

Plane wave solutions of (23) have the form

$$\vec{E}(r) = \vec{E}_0 e^{-i\vec{k} \cdot \vec{x}}, \tag{24}$$

Then, since

$$\frac{\partial}{\partial x_i} e^{-i\vec{k} \cdot \vec{x}} = -ik_i e^{-i\vec{k} \cdot \vec{x}}, \tag{25}$$

it follows that

$$\left(\vec{\nabla}\,\vec{\nabla} - \Delta\,\tilde{1} - \frac{\omega^2}{c^2}\frac{\tilde{\epsilon}}{\epsilon_0}\right) e^{-i\vec{k} \cdot \vec{x}} = -\tilde{\Omega}(\vec{k}) e^{-i\vec{k} \cdot \vec{x}}, \tag{26}$$

where

$$\tilde{\Omega}(\vec{k}) = \vec{k}\,\vec{k} - k^2\tilde{1} + \frac{\omega^2}{c^2}\frac{\tilde{\epsilon}}{\epsilon_0} \tag{27}$$

is the "wave matrix." The homogeneous wave equation thus has the form

$$\tilde{\Omega}(\vec{k})\vec{E}_0 e^{-i\vec{k}\cdot\vec{x}} = 0. \tag{28}$$

The condition for nonvanishing fields is

$$\det \tilde{\Omega}(\vec{k}) = 0, \tag{29}$$

where det denotes the determinant of the elements of the tensor. The planes of equal phase are specified by the wave normal $\vec{n}=(n_1, n_2, n_3)$ such that $\vec{k}=k\vec{n}$. The components of the direction cosines may be expressed in terms of angles α and β by

$$n_1 = \sin\alpha\cos\beta,$$

$$n_2 = \sin\alpha\sin\beta,$$

$$n_3 = \cos\alpha.$$

Using the specific form of the dielectric tensor defined above, it is not difficult to show that

$$\det\tilde{\Omega}(\vec{k}) = \mu_0\omega^2(\epsilon\sin^2\alpha + \hat{\epsilon}\cos^2\alpha)(k^2 - k_{\mathrm{I}}^2)(k^2 - k_{\mathrm{II}}^2) = 0, \tag{30}$$

where

$$\left.\begin{matrix} k_{\mathrm{I}}^2 \\ k_{\mathrm{II}}^2 \end{matrix}\right\} = \frac{(\epsilon^2 - g^2)\sin^2\alpha + \epsilon\hat{\epsilon}(1+\cos^2\alpha) \pm \Gamma}{2(c^2/\omega^2)(\epsilon\sin^2\alpha + \hat{\epsilon}\cos^2\alpha)\epsilon_0}, \tag{31}$$

where

$$\Gamma = [(\epsilon^2 - g^2 - \epsilon\hat{\epsilon})^2\sin^4\alpha + 4g^2\hat{\epsilon}^2\cos^2\alpha]^{1/2}. \tag{32}$$

The roots, k_{I} and k_{II}, of (30), are the (complex) wave numbers for the "ordinary" and the "extraordinary waves. They depend on the angle α between the wave normal and the d-c magnetic field.

Rather than using a Fourier representation to obtain the Green's function, it is instructive to utilize the properties of Dirac's δ functions. These are defined, in the usual manner, by

$$\int_{-\infty}^{+\infty} f(x')\delta(x - x')dx' = f(x) \tag{33}$$

and

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ik_1 x}dk_1. \tag{34}$$

Then, if

$$\vec{E}(r) = \int \tilde{G}(\vec{x}, \vec{x'})\cdot\vec{J}(x')d^3x', \tag{35}$$

it follows that $\tilde{G}$ must satisfy

$$\left(\vec{\nabla}\,\vec{\nabla} - \Delta\tilde{1} - \frac{\omega^2}{c^2}\frac{\tilde{\epsilon}}{\epsilon_0}\right)\tilde{G}(\vec{x},\vec{x'}) = -i\mu_0\omega\tilde{1}\delta(\vec{x}-\vec{x'}), \tag{36}$$

where

$$\delta(\vec{x}) = \delta(x_1)\delta(x_2)\delta(x_3). \tag{37}$$

This step may be readily verified if both sides of the equation are multiplied by $\vec{J}$ and then performing the volume integration. It should be noted that $\vec{\nabla}\,\vec{\nabla}$ and Δ operate on the variables $\vec{x}$ and not $\vec{x'}$.

To complete the derivation, (26) is rewritten in the form

$$\left(\vec{\nabla}\,\vec{\nabla} - \Delta\tilde{1} - \frac{\omega^2}{c^2}\right)\tilde{\Omega}^{-1}e^{-i\vec{k}\cdot\vec{x}} = -\tilde{1}e^{-i\vec{k}\cdot\vec{x}}, \tag{38}$$

where $\tilde{\Omega}^{-1}$ is the inverse of the "wave matrix." Both sides of this equation are now multiplied by $\exp(i\vec{k}\cdot\vec{x'})$ and the integrations $\int d^3k = \int\int\int dk_1dk_2dk_3$ are performed. On identifying this equation with (36), it readily follows that

$$\tilde{G}(\vec{x},\vec{x'}) = \frac{i\mu_0\omega}{(2\pi)^3}\int \tilde{\Omega}^{-1}e^{-i\vec{k}\cdot(\vec{x}-\vec{x'})}d^3k, \tag{39}$$

which establishes the desired result. As indicated, this is a special case of (19) for the generally anisotropic medium. Equation (39) was also given by Bunkin [1957] in slightly modified form.

The explicit form for the inverse of $\tilde{\Omega}$ is obtained from Cramer's rule; thus

$$\tilde{\Omega}^{-1} = \frac{\tilde{\Lambda}}{\det\tilde{\Omega}}, \tag{40}$$

where the matrix $\tilde{\Lambda}$ is the "adjoint" of $\tilde{\Omega}$. For a gyroelectric medium of the kind described, it is found that

$$\tilde{\Lambda}(\vec{k}) = k^2\vec{k}\,\vec{k} = k^2\frac{\omega^2}{c^2}\tilde{\mathscr{J}} + \frac{\omega^4}{c^4}\tilde{\mathscr{E}}, \tag{41}$$

where $\tilde{\mathscr{E}}$ is the adjoint of $\tilde{\epsilon}$, given explicitly by

$$\tilde{\mathscr{E}}\epsilon_0^2 = \begin{pmatrix} \epsilon\hat{\epsilon} & ig\hat{\epsilon} & 0 \\ -ig\hat{\epsilon} & \epsilon\hat{\epsilon} & 0 \\ 0 & 0 & \epsilon^2 - g^2 \end{pmatrix} \tag{42}$$

and $\tilde{\mathscr{J}}\epsilon_0 =$

$$\begin{pmatrix} \epsilon(n_1^2+n_2^2)+\hat{\epsilon}(n_1^2+n_3^2) & -ig(n_1^2+n_2^2)+\hat{\epsilon}n_1n_2 & \epsilon n_1n_3 - ign_2n_3 \\ ig(n_1^2+n_2^2)+\hat{\epsilon}n_1n_2 & \epsilon(n_1^2+n_2^2)+\hat{\epsilon}(n_2^2+n_3^2) & \epsilon n_2n_3 + ign_1n_3 \\ \epsilon n_1n_3 + ign_2n_3 & \epsilon n_2n_3 - ign_1n_3 & \epsilon(1+n_3^2) \end{pmatrix} \tag{43}$$

4. Modified Field Solution

A three-dimensional transform technique for solving the dipole problem has also been given by Mittra and Deschamps [1963]. Their results have a number of unique features which should be mentioned. In particular, some ingenious manipulations are made in order to carry out the inversion of the transforms. Furthermore, their final expressions are not restricted to the far zone or to a lossless medium.

The starting point, in Mittra's and Deschamps' analysis, is Maxwell's equations written for impressed electric current sources $\vec{J}$ and the tensor dielectric constant $\bar{\epsilon}$ as given in (22) while the permeability μ_0 is assumed to be that of free space. The eliminating $\vec{H}$ from Maxwell's equations leads readily to the usual inhomogeneous wave equation for $\vec{E}$ given by

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} - \mu_0 \omega^2 \bar{\epsilon} \vec{E} = -i\mu_0 \omega \vec{J}. \tag{44}$$

A coordinate system is now introduced which diagonalizes the tensor $\bar{\epsilon}$. The unit vectors in this system are denoted $\hat{U}_1$, $\hat{U}_2$, and $\hat{U}_3$, and are defined by

$$\hat{U}_1 = 2^{-1/2}(\hat{x} + i\hat{y}),\ \hat{U}_2 = 2^{-1/2}(\hat{x} - i\hat{y}),\ \text{and}\ \hat{U}_3 = \hat{z}, \tag{45}$$

in terms of the unit vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$ in the conventional Cartesian system. It is understood that the d-c magnetic field remains oriented in the axial $\hat{z}$ or $\hat{U}_3$ direction. An arbitrary vector $\vec{F}$ may be written in the U-system as

$$\vec{F} = \hat{U}_1 F_1 + \hat{U}_2 F_2 + \hat{U}_3 F_3, \tag{46}$$

where

$$F_1 = \vec{F} \cdot \hat{U}_2,\ F_2 = \vec{F} \cdot \hat{U}_1,\ \text{and}\ F_3 = \vec{F} \cdot \vec{U}_3. \tag{47}$$

In terms of the components F_x, F_y, and F_z in the conventional Cartesian system, it is seen that

$$F_1 = 2^{-1/2}(F_x - iF_y),\ F_2 = 2^{-1/2}(F_x + iF_y),\ \text{and}\ F_3 = F_z. \tag{48}$$

The diagonal nature of the U-system is immediately evident from the identity

$$\bar{\epsilon} = \hat{U}_1 \hat{U}_1 \epsilon_1 + \hat{U}_2 \hat{U}_2 \epsilon_2 + \hat{U}_3 \hat{U}_3 \epsilon_3, \tag{49}$$

where

$$\epsilon_1 + \epsilon + g,\ \epsilon_2 = \epsilon - g,\ \text{and}\ \epsilon_3 = \hat{\epsilon}.$$

The corresponding expression for the gradient operator in the U-system is

$$\nabla = \hat{U}_1 d_1 + \hat{U}_2 d_2 + \hat{U}_3 d_3, \tag{50}$$

where the derivative operators are defined by

$$d_1 = 2^{-1/2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),\ d_2 = 2^{-1/2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),\ \text{and}\ d_3 = \frac{\partial}{\partial z}. \tag{51}$$

After some algebraic manipulations, the inhomogeneous wave equation for $\vec{E}$ may be written in

convenient matrix form. Thus, in terms of the diagonalizing coordinate system,

$$\left\{\begin{pmatrix} -(d_1d_2+d_3^2) & d_1^2 & d_1d_3 \\ d_1^2 & -(d_1d_2+d_3^2) & d_2d_3 \\ d_3d_2 & d_1d_3 & -2d_1d_2 \end{pmatrix} - \mu_0\omega^2 \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}\right\}\begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = -i\mu_0\omega \begin{bmatrix} J_1 \\ J_2 \\ J_3 \end{bmatrix}. \quad (52)$$

These are matrix equations for the fields, first given in this form by Mittra and Deschamps [1963]. To solve these, it is necessary to employ Fourier representations which transform the differential operators d_1, d_2, and d_3 to algebraic quantities.

The three-dimensional Fourier transform of $E_i(i=1, 2, 3)$, denoted $\mathscr{E}_i(i=1, 2, 3)$, is defined by

$$\mathscr{E}_i = (2\pi)^{-3}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E_i(x, y, z)\exp\left[+i(k_xx+k_yy+k_zz)\right]dxdydz\ . \quad (53)$$

The differential operators now transform in the following manner

$$d_1 \rightarrow 2^{-1/2}i(k_x-ik_y),\ d_2 \rightarrow 2^{-1/2}i(k_x+ik_y),\ d_3 \rightarrow ik_z.$$

For convenience of field calculations, it is desirable to introduce the usual polar form defined by $k_x = k\sin\alpha\cos\beta$, $k_y = k\sin\alpha\sin\beta$, and $k_z = k\cos\alpha$, whereupon

$$d_1 \rightarrow i2^{-1/2}k(\sin\alpha)e^{-i\beta},\ d_2 \rightarrow i2^{-1/2}k(\sin\alpha)e^{i\beta},\ d_3 \rightarrow ik\cos\alpha.$$

The transform of the matrix equation (52) now has the form

$$\left\{-\mu_0\omega^2\begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} + k^2\begin{pmatrix} -\left(\frac{s^2}{2}+c^2\right) & \frac{1}{2}s^2e^{-2i\beta} & 2^{-1/2}sce^{-i\beta} \\ \frac{1}{2}s^2e^{2i\beta} & -\left(\frac{s^2}{2}+c^2\right) & 2^{-1/2}sce^{+i\beta} \\ 2^{-1/2}sce^{+i\beta} & 2^{-1/2}sce^{-i\beta} & -s^2 \end{pmatrix}\right\}\begin{bmatrix} \mathscr{E}_1 \\ \mathscr{E}_2 \\ \mathscr{E}_3 \end{bmatrix} = -i\mu_0\omega\begin{bmatrix} \mathscr{J}_1 \\ \mathscr{J}_2 \\ \mathscr{J}_3 \end{bmatrix}, \quad (54)$$

where $s = \sin\alpha$ and $c = \cos\alpha$.

The determinant Δ of the coefficients inside the curly brackets of (54) is seen to be of the form

$$\Delta = \mu_0\omega^2(\epsilon\sin^2\alpha + \hat{\epsilon}\cos^2\alpha)(k^2-k_{\mathrm{I}}^2)(k^2-k_{\mathrm{II}}^2), \quad (55)$$

which is identical to (30) as it must be. The roots of the equation $\Delta = 0$ are k_{I} and k_{II} and they are given explicitly by (31). As indicated previously, k_{I}/k_0 and k_{II}/k_0 are the indices of refraction for plane ordinary and extraordinary waves, respectively.

Since the matrix equation (54) involves only algebraic quantities the solution may be obtained directly. Thus, in matrix notation,

$$[\mathscr{E}] = -i\mu_0\omega[A][\mathscr{J}], \quad (56)$$

where the elements of the inverted matrix are given by

$$A_{mn} = a_{mn}/\Delta. \quad (57)$$

The coefficients a_{mn} are given by

$$a_{11} = (1/2)k^4s^2 - k^2\mu_0\omega^2\left\{\epsilon_3s^2 + \epsilon_3\left(\frac{s^2}{2}+c^2\right)\right\} + \mu_0\omega^4\epsilon_3\epsilon_2,$$

$$a_{12} = e^{-2i\beta}(1/2)k^2s^2(k^2 - \mu_0\omega^2\epsilon_3),$$

$$a_{13} = e^{-i\beta}2^{-1/2}k^2sc(k^2 - \mu_0\omega^2\epsilon_2),$$

$$a_{21} = e^{2i\beta}(1/2)k^2s^2(k^2 - \mu_0\omega^2\epsilon_3),$$

$$a_{22} = (1/2)k^4s^2 - k^2\mu_0\omega^2\left[\epsilon_1 s^2 + \epsilon_3\left(\frac{s^2}{2} + c^2\right)\right] + \mu_0^2\omega^4\epsilon_1\epsilon_3,$$

$$a_{23} = e^{i\beta}2^{-1/2}k^2sc(k^2 - \mu_0\omega^2\epsilon_1),$$

$$a_{31} = e^{i\beta}2^{-1/2}k^2sc(k^2 - \mu_0\omega^2\epsilon_2),$$

$$a_{32} = 2^{-1/2}e^{-i\beta}k^2sc(k^2 - \mu_0\omega^2\epsilon_1),$$

$$a_{33} = k^4c^2 - \mu_0\omega^2k^2(\epsilon_2 + \epsilon_1)\left(\frac{s^2}{2} + c^2\right) + \mu_0^2\omega^4\epsilon_1\epsilon_2.$$

The explicit evaluation of the three-dimensional inverse transforms remains to be done. The usual approach is to carry out a stationary phase type calculation which is stated to be valid in the far zone. This aspect of the subject is discussed below, but first, it is appropriate to describe the approach by Mittra and Deschamps [1963], which is not restricted to the far field.

To be explicit, an electric dipole source is located at the origin and, to achieve simplicity, the dipole is oriented in the z-direction which is parallel to the d-c magnetic field. Thus

$$\vec{J} = p_0\hat{z}\delta(x)\delta(y)\delta(z), \tag{58}$$

where $\hat{z}$ is a unit vector in the z (or U_3) direction. Thus, from (55), (56), and (57),

$$\mathscr{E}_n = -i\mu_0\omega p_0\frac{a_{n3}(k, \alpha, \beta)}{\Delta(k, \alpha)} \text{ for } n = 1, 2, 3, \tag{59}$$

and the inverse transform may be written in the form

$$E_n = p_0\int_0^\infty\int_0^\pi\int_0^{2\pi}\mathscr{E}_n e^{-ikR[\sin\theta\sin\alpha\cos(\beta-\phi)+\cos\theta\cos\alpha]}\times k^2\sin\alpha d\beta d\alpha dk, \tag{60}$$

where (R, θ, ϕ) are the spherical coordinates of the observer.

In the special case $n = 1$, the β integration is effected immediately since a_{33} is independent of β. Thus, it follows that

$$E_1 = \frac{i\mu_0\omega p_0}{k_0^2(2\pi)^{-1}}\nabla^2 I(R, \theta), \tag{61}$$

where

$$I(R, \theta) = k_0^2\int_0^\infty\int_0^\pi\frac{a_{33}}{\Delta}e^{-ikR\cos\theta\cos\alpha}J_0(kR\sin\theta\sin\alpha)\sin\alpha d\alpha dk. \tag{62}$$

To evaluate this integral, Mittra and Deschamps [1963] recognize that I may be written as a sum of a singular part I_s and a finite part I_f. That is, I_s is of order $(1/R)$ for R tending to zero, whereas I_f is finite for all R and θ. Following some ingenious manipulations, they find that

$$\nabla^2 I_s = -\frac{\pi}{2}\left(\frac{\epsilon_0}{\epsilon}\right)^{1/2}\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\frac{1}{R'}, \tag{63}$$

where $R' = \left[\frac{\epsilon}{\epsilon_0} z^2 + \frac{\hat{\epsilon}}{\epsilon_0} \rho^2\right]^{1/2}$. This result has the correct singularity and, in addition, it demonstrates that I_s is independent of g, the off-diagonal factor in the expression for the tensor dielectric constant (e.g., (22)).

The evaluation of $\nabla^2 I_f$ appears to be more complicated. It turns out that while I_f is finite at the origin, $\nabla^2 I_f$ does have a singularity of the type $O(1/R)$. Mittra and Deschamps [1963] have given an explicit expression for $\nabla^2 I_f$ but it involves double integrals which, fortunately, have finite limits. Apparently, a great deal of numerical work is required in order to obtain useful numerical results from this approach. It is rather surprising that they do not carry out the α integration explicitly by using contour integration methods. This requires that one evaluate the residues of the poles of the integrand which occur at the zeroes of Δ for a fixed value of k. This operation will not be carried through since it leads directly to the (single) integral representation, given by Barsukov [1959] and Arbel and Felsen [1963], which may be obtained in a more direct fashion as outlined below. The latter approach is particularly appropriate when dealing with horizontally stratified media.

5. Sources in Stratified Anisotropic Media

Before tackling the generally stratified medium, the appropriate expressions for the (primary) field of a point source in a homogeneous medium are derived. With respect to the Cartesian coordinate system, an electric dipole is located at $(0, 0, z_0)$. The d-c magnetic field is oriented along the z axis and thus the dielectric dyadic $\hat{\epsilon}$ has the tensor form.

$$(\epsilon) = \begin{pmatrix} \epsilon & -ig & 0 \\ ig & \epsilon & 0 \\ 0 & 0 & \hat{\epsilon} \end{pmatrix}, \tag{64}$$

where ϵ, $\hat{\epsilon}$, and g are specified constants for each homogeneous region. For convenience, Maxwell's equations

$$\vec{\nabla} \times \vec{H} = i\omega \bar{\epsilon} \cdot \vec{E} + \vec{J} \tag{65}$$

and

$$\nabla \times \vec{E} = -i\mu_0 \omega \vec{H} \tag{66}$$

are supplemented by the divergence relations

$$\nabla \cdot (\bar{\epsilon} \cdot \vec{E}) = \bar{\rho} = \frac{i}{\omega} \vec{\nabla} \cdot \vec{J} \tag{67}$$

and

$$\nabla \cdot \vec{H} = 0\,, \tag{68}$$

where $\bar{\rho}$ is the charge density.

The preceding equations may be written out explicitly in terms of cylindrical coordinate systems. Then, it is a straightforward matter to eliminate the E_r, E_ϕ, H_r, and H_ϕ components. It is found that

$$\nabla_t^2 E_z + \frac{\hat{\epsilon}}{\epsilon}\frac{\partial^2 E_z}{\partial z^2} + \frac{k_0^2 \hat{\epsilon} E_z}{\epsilon_0} - \frac{\mu_0 \omega g}{\epsilon}\frac{\partial H_z}{\partial z} = i\mu_0 \omega J_z + \frac{1}{\epsilon}\frac{\partial \bar{\rho}}{\partial z} \tag{69}$$

and

$$\nabla_t^2 H_z + \frac{\partial^2 H_z}{\partial z^2} + \frac{k_0^2}{\epsilon\epsilon_0}(\epsilon^2 - g^2)H_z + \frac{\hat{\epsilon}}{\epsilon}\omega g \frac{\partial E_z}{\partial z} = -[\vec{\nabla} \times \vec{J}]_z + \frac{\omega g \bar{\rho}}{\epsilon}, \tag{70}$$

where

$$[\vec{\nabla} \times \vec{J}]_z = \frac{\partial J_\phi}{\partial \rho} - \frac{1}{\rho}\frac{\partial J_\rho}{\partial \phi}, \tag{71}$$

and

$$\nabla_t^2 = \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial}{\partial^2 \phi} \tag{72}$$

is the transverse Laplacian operator. In this section $k_0 = \omega/c = 2\pi/\text{wavelength}$ and ϵ_0 is the dielectric constant of free space.

The above coupled equations for E_z and H_z will hold for any (piecewise) homogeneous region. A subscript m is added to the field quantities when reference is to be made to the mth homogeneous region. The integer m is allowed to range from 1 to M. The source dipole is taken to be in the pth layer where p is an integer equal to or between 1 and M.

To facilitate matching of boundary conditions, it is convenient to employ the following spectral representations of Dirac delta functions:

$$\delta(z - z_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{iu(z-z_0)}du \tag{73}$$

and

$$\frac{\delta(\rho)}{\rho} = \int_0^\infty J_0(\lambda\rho)\lambda d\lambda. \tag{74}$$

Therefore, if the dipole of moment p_0, located at $z = z_0$, is oriented in the z direction the current density has the following form:

$$J_z = i\omega p_0 \delta(z - z_0)\delta(\rho)/(2\pi\rho) = \frac{i\omega p_0}{(2\pi)^2}\int_0^\infty \int_{-\infty}^{+\infty} J_0(\lambda\rho)e^{iu(z-z_0)}\lambda u d\lambda du. \tag{75}$$

The corresponding formula for the charge density is

$$\bar{\rho} = -\frac{ip_0}{(2\pi)^2}\int_0^\infty \int_{-\infty}^{+\infty} J_0(\lambda\rho)e^{iu(z-z_0)}\lambda u d\lambda du. \tag{76}$$

In the sth region, which contains the source, the axial field components are written in the form

$$E_z = E_z^p + E_z^s \tag{77}$$

and

$$H_z = H_z^p + H_z^s, \tag{78}$$

where the primary and secondary fields are indicated by a superscript p and s, respectively. It is understood that, if the sth region were unbounded, the secondary fields would not be present.

Taking a clue from the appearance of (75) and (76), the axial components of the primary field are written in the form

$$E_z^p = \frac{p_0}{(2\pi)^2}\int_0^\infty \int_{-\infty}^{+\infty} A(\lambda, u)J_0(\lambda\rho)e^{iu(z-z_0)}d\lambda du \tag{79}$$

and

$$H_z^p = \frac{p_0}{(2\pi)^2}\int_0^\infty \int_{-\infty}^{+\infty} B(\lambda, u)J_0(\lambda\rho)e^{iu(z-z_0)}d\lambda du. \tag{80}$$

Substituting (75), (76), (79), and (80) into (69) and (70), explicit expressions may be found for the functions A and B. The integrations, with respect to u, may be easily carried out and the final result is

$$E_z^p = \frac{\epsilon}{\hat{\epsilon}^2}\, p_0 \int_0^\infty [u_1(u_1^2+s_1^2)e^{-u_1|z-z_0|} - u_2(u_2^2+s_1^2)e^{-u_2|z-z_0|}] \times \frac{J_0(\lambda\rho)}{s_2^2(u_2^2-u_1^2)}\, \lambda^3 d\lambda \tag{81}$$

and

$$H_z^p = -\frac{\omega g}{\hat{\epsilon}}\, p_0 \frac{(z-z_0)}{|z-z_0|} \int_0^\infty [e^{-u_1|z-z_0|} - e^{-u_2|z-z_0|}] \times \frac{J_0(\lambda\rho)\lambda^3 d\lambda}{u_2^2-u_1^2}, \tag{82}$$

where

$$s_1^2 = k_0^2(\epsilon/\epsilon_0) - \lambda^2,\ s_2^2 = (\epsilon/\hat{\epsilon})[k_0^2(\hat{\epsilon}/\epsilon_0) - \lambda^2], \tag{83}$$

where u_1 and u_2 are roots of the equation

$$u^4 + u^2(s_1^2+s_2^2) + s_1^2 s^2 - s_2^2 \frac{k_0^2 g^2}{\epsilon\epsilon_0} = 0 \tag{84}$$

chosen so that Re $u_1 > 0$ and Re $u_2 > 0$.

Expressions for the other components of the primary field may be readily found from the axial components given here.

Expressions for the secondary field, in the sth region, are now sought in the form

$$E_z^s = \frac{\epsilon}{\hat{\epsilon}^2} p_0 \int_0^\infty \{u_1(u_1^2+s_1^2)[\varphi_1(\lambda)e^{-u_1 z} + \psi_1(\lambda)e^{+u_1 z}]$$

$$+ u_2(u_2^2+s_1^2)[\varphi_2(\lambda)e^{-u_2 z} + \psi_2(\lambda)e^{+u_2 z}]\} \frac{J_0(\lambda\rho)\lambda^3 d\lambda}{s_2^2(u_2^2-u_1^2)} \tag{85}$$

and

$$H_z^s = -\frac{\omega g}{\hat{\epsilon}}\, p_0 \int_0^\infty \{[\varphi_1(\lambda)e^{-u_1 z} - \psi_1(\lambda)e^{+u_1 z}] - [\varphi_2(\lambda)e^{-u_2 z} - \psi_2(\lambda)e^{+u_2 z}]\} \frac{J_0(\lambda\rho)\lambda^3 d\lambda}{(u_2^2-u_1^2)}, \tag{86}$$

where the functions $\varphi_1(\lambda)$, $\varphi_2(\lambda)$, $\psi_1(\lambda)$, and $\psi_2(\lambda)$ are undetermined.

The axial field components, E_z^n and H_z^n, in the other homogeneous slabs would have the same form as the above expressions if the appropriate material constants of the medium are employed (e.g., for the nth slab ϵ, $\hat{\epsilon}$, and g are replaced by ϵ_n, $\hat{\epsilon}_n$, and g_n, respectively). In each slab, there are four unknown functions (i.e., φ_1^n, φ_2^n, ψ_1^n, and ψ_2^n), and these may be found from the matching of the four tangential field components at each interface. In addition, it is required that only outgoing waves are permitted at $z = \pm\infty$. Thus, in principle, the complete formal solution is available.

Using the method outlined here, Barsukov [1959] has given an explicit expression for the field of a vertical electrical dipole at $z = z_0$, which is immersed in a semi-infinite anisotropic medium ($z > 0$). In the region below (i.e., $z < 0$), the medium is assumed to be free space with electrical constants ϵ_0 and μ_0.

For $z > 0$, the axial fields have the form

$$E_z = E_z^p + E_z^s, \tag{87}$$

$$H_z = H_z^p + H_z^s. \tag{88}$$

Here, E_z^p and H_z^p are the primary fields given by (81) and (82), while E_z^s and H_z^s are the secondary

fields which have the form of (85) and (86) with the condition that $\psi_1(\lambda)$ and $\psi_2(\lambda)$ are both zero.

For the isotropic region $z<0$, it is convenient to represent the field in terms of electric and magnetic Hertz vectors, $\vec{\Pi}_e$ and $\vec{\Pi}_m$. Thus

$$\vec{E}=k_0^2\vec{\Pi}_e+\nabla(\nabla\cdot\vec{\Pi}_e)-i\mu_0\omega\nabla\times\vec{\Pi}_m, \tag{89}$$

$$\vec{H}=i\epsilon_0\omega\nabla\times\vec{\Pi}_e+k_0^2\vec{\Pi}_m+i\epsilon_0\omega\nabla\times\vec{\Pi}_e, \tag{90}$$

where

$$\vec{\Pi}_e=(0,\ 0,\ \Pi_e) \text{ and } \vec{\Pi}_m=(0,\ 0,\ \Pi_m).$$

The following representations are employed:

$$\Pi_e=\frac{p_0}{k_0}\int_0^\infty F_1(\lambda)J_0(\lambda\rho)e^{iu_0z}\lambda d\lambda \tag{91}$$

and

$$\Pi_m=\frac{ip_0}{k_0}\int_0^\infty F_2(\lambda)J_0(\lambda\rho)e^{iu_0z}\lambda d\lambda, \tag{92}$$

where $u_0=(\lambda^2-k_0^2)^{1/2}$ with $\mathrm{Re}\ u_0>0$. The scalar functions, Π_e and Π_m, satisfy the wave equation $(\nabla^2+k_0^2)\Pi_{e,m}=0$; they behave as outgoing waves at $z=-\infty$, and they are independent of the azimuthal coordinate ϕ. The functions $\varphi_1(\lambda)$, $\varphi_2(\lambda)$, $F_1(\lambda)$, and $F_2(\lambda)$ can be determined explicitly from the four boundary conditions (requiring the continuity of the tangential fields) at $z=0$. Barsukov [1959] gives

$$\varphi_1(u_1,\ u_2)=-\frac{1}{\delta(u_1,\ u_2)}\left[\delta(-u_1,u_2)e^{-u_1z_0}+2\frac{\epsilon}{\hat{\epsilon}}u_2\lambda^2(\hat{\epsilon}-\epsilon)(u_2^2+s_1^2)e^{-u_2z_0}\right]. \tag{93}$$

$$\varphi_2(u_1,\ u_2)=\varphi_1(u_2,\ u_1), \tag{94}$$

$$F_1(u_1,\ u_2)=\frac{2\epsilon k_0}{\hat{\epsilon}\delta(u_1,\ u_2)}\left[u_1(u_1^2+s_1^2)(u_0+u_2)e^{-u_1z_0}-u_2(u_2^2+s_1^2)(u_0+u_1)e^{-u_2z_0}\right], \tag{95}$$

$$F_2(u_1,\ u_2)=\frac{2igk_0^2}{\hat{\epsilon}\delta(u_1,\ u_2)}\left[u_1(\epsilon u_1u_2-\epsilon_0s_2^2)e^{-u_1z_0}-u_2(\epsilon u_1u_2-\epsilon_0s_2^2)e^{-u_2z_0}\right], \tag{96}$$

where

$$\delta(u_1,\ u_2)=(u_2-u_1)\ [\epsilon_0s_2^2(u_1u_2-s_1^2)-\epsilon u_0^2(u_1u_2-s_2^2)+u_0(u_1+u_2)(\epsilon_0s_2^2-\epsilon u_1u_2)]\ . \tag{97}$$

This constitutes the complete formal solution of the problem. Explicit expressions for the field components are obtained by differentiation according to (89), (90), (91), and (92).

The general results are rather complicated; however, some simplicity is achieved by considering the radiation fields in the lower half space. Physically, this corresponds to the situation where the observer is at a distance, below the interface, which is large compared with the wavelength. The integrals to contend with are of the form

$$\int_0^\infty A(\lambda)J_0(\lambda\rho)e^{-u|z_0|}\lambda d\lambda=I(\rho,\ z), \tag{98}$$

where $A(\lambda)$ is a slowly varying function of λ. A saddle point evaluation of the integral yields

$$I(\rho,\ z)\sim ik_0\cos\theta A(k_0\sin\theta)e^{-ik_0R}/R, \tag{99}$$

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where

$$R=(\rho^2+z^2)^{1/2} \text{ and } \tan\theta=-(\rho/z).$$

As indicated, the saddle point or principal value occurs at $\lambda=k_0 \sin\theta$ where θ is the angle subtended by the direction to the observer and the negative z-axis. This asymptotic result is valid provided $|k_0 z| >> 1$ and $|z| >> z_0$. The first inequality appears to be very important since if kz is small, the contributions from the poles of $A(\lambda)$ lead to surface waves which have a significant amplitude compared with the space wave. The pole wave contributions were not considered by Barsukov [1959].

6. Radiation Fields in Anisotropic Media

An important aspect of this subject is the nature of the radiation or far-zone fields when both source and observer are within the anisotropic medium. This particular problem has been considered in some detail by Arbel and Felsen [1963]. In their work they make extensive use of the geometrical properties of the refractive index surfaces. For purposes of this discussion, the medium will be considered homogeneous and unbounded. Furthermore, the source is an electric dipole of moment p_0 located at the origin of a cylindrical coordinate system (ρ, ϕ, z) and oriented in the z direction. The applied d-c magnetic field is also directed in the z direction. The medium is nonmagnetic with permeability μ_0, while the electric properties are described in terms of a tensor dielectric constant $\bar{\epsilon}$ of the form given by (22). Losses in the medium are assumed to be zero. The Fourier-Bessel representations of the fields given by Arbel and Felsen [1963] are closely related to (81) and (82) first given by Barsukov [1959]. The essential modifications to (81) and (82) are to replace the variable λ by $k_0 S$ where S is now a dimensionless variable and $k_0=\omega/c$. By using identities $J_0(x)=\frac{1}{2}[H_0^{(1)}(x)+H_0^{(2)}(x)]$ and $H_0^{(2)}(x)=-H_0^{(1)}(-x)$, the integration contour may be changed from along the positive real axis to a straight line running from $S=-\infty$ to $+\infty$. Because of a branch point singularity at $S=0$, the integration path is indented around this point by a small semi-circle in the lower half plane of the complex S plane. Additional substitutions in (81) and (82) are to replace the axial propagation constants u, u_1, and u_2 by ik_0C, ik_0C_1, and ik_0C_2, respectively. The subscripts 1 and 2 may be identified with the ordinary and the extraordinary waves in the classical magneto-ionic theory.

The resulting vector electric fields may be written in the form

$$\vec{E}=\vec{E_1}+\vec{E_2}$$

where the field components are given by

$$(E_\rho)_{1,2}=iA\frac{z}{|z|}\int_{-\infty}^{+\infty} S^2\frac{S^2\mp\Delta}{\pm\Delta}H_1^{(2)}(k_0S\rho)\exp\left[-ik_0C_{1,2}|z|\right]dS, \tag{100}$$

$$(E_\phi)_{1,2}=2A\frac{\epsilon'}{\epsilon-1}\int_{-\infty}^{+\infty}\frac{S^2}{\pm\Delta}H_1^{(2)}(k_0S\rho)\left[-ik_0C_{1,2}|z|\right]dS, \tag{101}$$

and

$$(E_z)_{1,2}=A\int_{-\infty}^{+\infty}\frac{S^2\mp\Delta}{\pm\Delta}\frac{C_{1,2}S^3}{1-S^2}H_0^{(2)}(kS_\rho)\exp\left[-ik_0C_{1,2}|z|\right]dS, \tag{102}$$

where

$$A=\mu_0\omega^2 p_0/(16\pi), \tag{103}$$

and

$$\Delta=\frac{\epsilon-\epsilon_0}{|\epsilon-\epsilon_0|}\left[S^4+4\left(\frac{g}{\epsilon-1}\right)^2(1-S^2)\right]^{1/2}. \tag{104}$$

The functions C_1 and C_2 are proportional to u_1 and u_2 as noted. The latter may be obtained from the roots of the quadratic equation in u^2 given by (84), which is equivalent to the Appleton-Hartree equation. Explicitly,

$$C_{1,2}=[U\pm(U^2-W)^{1/2}]^{1/2}=\left[U\pm\left(\frac{\epsilon}{\epsilon_0}-1\right)\frac{\Delta}{2}\right]^{1/2}, \tag{105}$$

where

$$U=\frac{\epsilon}{\epsilon_0}-\frac{(\epsilon+\epsilon_0)}{2\epsilon_0}S^2, \tag{106}$$

$$W=\frac{\epsilon}{\epsilon_0}(1-S^2)\left[\frac{\epsilon^2-g^2}{\epsilon\epsilon_0}-S^2\right]. \tag{107}$$

Arbel and Felsen [1963] have devoted a great deal of attention to the proper definition of the multivalued function C and the disposition of the integration path in relation to the singularities of the integrands in (100), (101), and (102). In the first place, it is to be observed that C_1 and C_2 are associated respectively with the + and − signs in (105), and that $(U^2-W)^{1/2}$ is defined to be positive when real. The definition of C_1 given above implies that C_1 has branch-point singularities at $U^2=W$ (or at $\Delta=0$), and at those values of S for which $C_1=0$. Similarly, C_2 has branch-point singularities at $\Delta=0$, and at those values of S for which $C_2=0$. However, a series expansion of the total integrand about $\Delta=0$ contains only even powers of Δ, which means that $\Delta=0$ is a regular point. Thus, no special care need be taken in the definition of $C_{1,2}$ at $\Delta=0$ if the "ordinary" and "extraordinary" are combined and treated together. Also, it should be mentioned that the factor $(1-S^2)^{-1}$ in the integrand of (102) does not give rise to a pole contribution since it is canceled by other factors.

If C_1 or C_2 are real, the quantity $\exp(-ik_0C_{1,2}z)$ represents a propagating plane wave in the positive z-direction away from the source. On physical grounds, it may be supposed that each component plane wave, in the total spectrum, has a positive component in the z-direction. This radiation condition permits the unique determination of $C_{1,2}$ over the range of S for which the propagation constants are real. As Arbel and Felsen [1963] show, the contributions to the power flow, when the propagations are complex, are zero. The resulting "energy radiation condition" is that, when $C_{1,2}$ is real,

$$\left(\frac{\mu_0}{\epsilon_0}\right)^{1/2}Y_{1,2}=C_{1,2}\frac{4}{(\Delta^2\pm S^2\Delta)}\left(\frac{g}{\epsilon-1}\right)^2>0, \tag{108}$$

where Y_1 and Y_2 may be interpreted as admittances for ordinary and extraordinary waves traveling in the positive z-direction.

It is evident from the above inequality that the sign of $C_{1,2}$ is identical with that of $\Delta^2-S^2\Delta$. This condition for real $C_{1,2}$, when supplemented with the condition $\operatorname{Im} C_{1,2}<0$ for $C_{1,2}$ complex, may be used to determine the analytic continuation of the function $C_{1,2}(S)$ around branch-point singularities on the real axis of the S plane.

As indicated above, the propagating modes correspond to the situation where

$$C_{1,2}^2=U\pm(U^2-W)^{1/2} \tag{109}$$

is positive real. Thus, it is evident that ordinary and extraordinary modes propagate when $U>0$. On the other hand, if $(U^2-W)>|U|$, i.e., $W<0$, the ordinary mode propagates for $U\leqslant 0$, while the extraordinary mode does not propagate.

There is actually a very simple graphical procedure which utilizes the dispersion curves of $C(S)$ versus S. For example, the plane wave, characterized by the variation

$$\exp[-ik_0(S\rho+Cz)],$$

carries power in the direction perpendicular to the dispersion curve at the point S, C. The radiation condition corresponding to choosing those values of C_1 and C_2 for which the outward normal to the dispersion curve, has a positive component in the z-direction.

The integrals in (100), (101), and (102) may be evaluated asymptotically by the method of saddle points. This has been demonstrated by Arbel and Felsen [1963] in an elegant manner.

It is convenient to introduce spherical coordinates (r, θ) which are defined by $\rho = r \sin \theta$ and $z = r \cos \theta$. Then, under the assumption that $k_0 r \sin \theta >> 1$, the asymptotic approximation

$$H_n^{(2)}(k_0 S\rho) \sim \left(\frac{2}{\pi k_0 S r \sin \theta}\right)^{1/2} \exp\left[-ik_0 S r \sin \theta + in\pi/2 + i\pi/4\right] \tag{110}$$

is introduced. Thus, for most practical purposes, the integrals to contend with are of the form

$$B = \int_{-\infty}^{+\infty} L(S) \exp\left[-ik_0 r M(S)\right] dS, \tag{111}$$

where

$$M(S) = C(S)|\cos \theta| + S \sin \theta, \tag{112}$$

and $k_0 r$ is to be regarded as a large parameter.

The principal contributions to the integrand occur at the vicinity of the saddle points S^i determined by

$$M'(S^i) = 0, \tag{113}$$

or

$$C'(S^i) = -|\tan \theta|. \tag{114}$$

Propagating waves correspond to the saddle points where both $C(S^i)$ and S^i are real. If these quantities are complex, the associated fields are highly attenuated and need not concern us further.

The technique to evaluate the integral asymptotically consists of deforming the integration path to the steepest descent paths through each saddle point. The ends of these paths are connected up in the valley regions of the S plane where $-\text{Im } C(S)$ is a large positive quantity. Thus, because of the heavy exponential damping, these portions of the path contribute a negligible amount and may be ignored for the present analysis. In the vicinity of a real saddle point S_i, the factor $\exp\left[-ik_0 r C(S)\right]$ decays most rapidly along the steepest descent path which is inclined at angles of $\pm 45°$ with the real axis.

The asymptotic representation for the integral is then found to be of the form

$$B \cong \sum_i B^{(i)}, \tag{115}$$

where the contribution from an individual saddle point is

$$B^{(i)} \sim L(S^i) \left[\frac{2\pi}{k_0 r |M''(S^i)|}\right]^{1/2} e^{\mp i\pi/4} e^{-ik_0 r M(S^i)}, \tag{116}$$

where the argument $-i\pi/4$ is used when $M''(S^i) > 0$, and $+i\pi/4$ is used when $M''(S^i) < 0$. It is assumed that the integrals are of the form such that the saddle points have alternating signs along the real axis of S.

To express the final results for the radiation fields, the unit vectors, $\hat{\rho}$, $\hat{\phi}$, and $\hat{z}$ are introduced along with the polarization vectors

$$\vec{p^i} = \hat{\rho} S^i + \hat{z} C^i \text{ and } \vec{q_i} = \hat{\rho} S^i + \hat{z} \frac{1 - (S^i)^2}{C^i}. \tag{117}$$

Then, following Arbel and Felsen [1963], it is found that the asymptotic versions of (100), (101), and (102) may be written

$$\hat{\rho}E_\rho + \hat{z}E_z \sim 2iA \sum_i \beta^i (R^i)^{1/2} G^i \frac{e^{-ik_0 r N^i(\theta)}}{k_0 r} (\vec{q} \times \hat{\phi}), \tag{118}$$

and

$$E_\phi \sim 2A \sum_i \beta^i (R^i)^{1/2} F^i \frac{e^{-ik_0 r N^i(\theta)}}{k_0 r}, \tag{119}$$

where $N_i(\theta)$ is the "ray refractive index" defined by

$$M(S^i) = N^i(\theta). \tag{120}$$

The radius of curvature R^i of the dispersion curve (i.e., $C(S)$ versus S for real S) at the saddle point S^i is obtained from

$$R^i = \frac{\{1 + [C'(S^i)]^2\}^{3/2}}{|C''(S^i)|} = \{|C''(S^i)| \cos^3 \theta\}^{-1}. \tag{121}$$

In (118) and (119) $\beta^i = 1$ if $M''(S^i) < 0$ and $\beta^i = i$ if $M''(S^i) > 0$. Finally, the amplitude factors are obtained from

$$F^i = \left(\frac{S_i}{\sin \theta}\right) \frac{2S^i}{\Omega} \left(\frac{g}{\epsilon - \epsilon_0}\right) \cos \theta, \tag{122}$$

and

$$G^i = \pm \left(\frac{S_i}{\sin \theta}\right)^{1/2} \frac{C^i S^i}{1 - (S^i)^2} \frac{(S^i)^2 - \Omega}{\Omega} \cos \theta, \tag{123}$$

where

$$\Omega = \frac{2\left[C^2(S^i) - \frac{\epsilon}{\epsilon_0} + \frac{\epsilon + \epsilon_0}{2\epsilon_0} (S^i)^2\right]}{\left(\frac{\epsilon}{\epsilon_0} - 1\right)}. \tag{124}$$

It is understood that (118) and (119) both hold individually for ordinary and extraordinary waves. The summations extend over all contributing saddle points in each case. The exponential factors involve the ray refractive index $N^i(\theta)$ which is related to the polarization vector $p_i = \hat{\rho}S^i + \hat{z}C^i$ by the scalar identity $N^i(\theta) = \vec{p} \cdot \vec{r_0}$ where $\vec{r_0}$ is a unit vector parallel to the ray direction which points from the source to the observer.

For the results given above, it is specifically assumed that $\hat{\epsilon}/\epsilon_0 > 0$. If $\hat{\epsilon}/\epsilon_0 < 0$, then the relevant field expressions are obtained by making the substitutions

$$(\hat{\epsilon})^{1/2} \to -i(-\hat{\epsilon})^{1/2}, \; C \to iC, \text{ and } S \to iS.$$

The characteristics of the radiation field in an anisotropic medium are critically dependent on the shape of the dispersion curve (or refractive index surface). It is evident that the number of contributing rays and their amplitude may be determined by a careful examination of the dispersion curve. The presence of open branches on the dispersion curve indicates presence of shadows where no rays may propagate. However, the field in the shadow region may be finite as a result of diffraction [Brekhovskikh, 1960]. This aspect of the subject is outside the scope of the present survey. Also, the presence of points of inflection in the dispersion curve (i.e., $C''(S^i) \cong 0$) will lead

to focusing effects which may be described by a modified saddle point procedure involving Airy functions [Wait, 1962]. This topic is also outside the scope of this discussion. However, the interested reader may find an elegant treatment of this particular problem in the paper by Arbel and Felsen [1963].

7. Final Remarks

The present expository paper has been a self-contained treatment of the theory of radiation from sources in cold magnetoplasma media.

In this review, no mention has been made of antenna impedance for anisotropic media. While this would seem to be a natural topic for this discussion, it has been deferred until a later time. One reason for this is the need to consider thermal effects and sheath phenomena, which are important near the source. Consideration resulting from the finite temperature leads to contributions from the acoustic-type waves not usually considered part of magneto-ionic theory.

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JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
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Publications of the National Bureau of Standards*

Selected Abstracts

An iterative unfolding procedure, R. P. Uhlig, *J. Res. NBS* **68A** (*Phys. and Chem.*), *No. 4, 401–407 (July-Aug. 1964*).
An iterative procedure for unfolding the affects of the finite resolution of a detector from an observed pulse height distribution discussed. The process is demonstrated for a particular detection system. Convergence and uniqueness properties of the method are discussed empirically.
A general expression for the propagated error resulting from errors in the detected pulse height distribution is derived. Approximations are made in order to evaluate the propagated error for a particular detector. These approximations become better as the resolution of the detector improves. The results indicate that the error rapidly approaches a limit of from 1.5 to 3 times the error in the observed distribution. This limit is reached in approximately three iterations.

On the relaxation of the hard-sphere Rayleigh and Lorentz gas, K. Anderson and K. E. Shuler, *J. Chem. Phys.* **40,** *No. 3, 633–650 (Feb. 1, 1964*).
As part of a study of the relaxation of non-equilibrium systems, the (translational) relaxation of a hard sphere Rayleigh and Lorentz gas is investigated. A master equation is formulated to represent the time variation of the distribution function of the subsystem particles and a technique is developed for transforming this integral master equation into differential Fokker-Planck equations. The Fokker-Planck equation for the Rayleigh gas is solved analytically and explicit solutions are presented for the relaxation of initial Maxwell and initial δ-function distributions of the energy. The Fokker-Planck equation for the hard sphere Lorentz gas does not appear to be susceptible to an analytical solution in terms of orthogonal polynomials, but some general properties of the solution have been established.

Review of mode theory of radio propagation in terrestrial waveguides, J. R. Wait, *Rev. Geophys.* **1,** *No. 4, 481–505 (Nov. 1963*).
This paper is an expository review of the theory of guided waves which occur in the earth's atmosphere. The subject is introduced by treating the problem of radio propagation around the surface of an airless spherical earth. This leads readily to the classical solutions of van der Pol and Bremmer and the more recent work of Fock in the U.S.S.R. The influence of a troposphere with a smooth profile of refractive index is then considered. This analysis, which follows the recent work of Armand in the U.S.S.R., confirms that amonotonically varying or smooth profile will not change the basic structure of the diffraction field. The modifications resulting from the presence of a tropospheric duct are developed by using a parabolic profile of refractive index in the manner suggested by Fock, Weinstein, and Belkina in the U.S.S.R. It is shown that the dominant modes in this system have low attenuation. The second and major part of the paper is devoted to the theory of the mode propagation of VLF radio waves. Here the effective waveguide is the space formed by the earth's surface and the lower edge of the ionosphere. The mode solution is developed as a natural generalization of the classical airless-earth theory following an earlier analysis by the author. It is shown, for frequencies less than about 10 kc/s, that the field may be described in terms of flat-earth modes which are analogous to those in a straight rectangular microwave guide. However, at higher frequencies the earth curvature plays a major role and the character and excitation of the modes are changed drastically. Complications resulting from the anisotropy of the ionosphere are also considered. A critical discussion ot the recent work on the subject is given.

Normal congruence subgroups of the modular group, M. Newman, *Am. J. Math.* **LXXXV,** *No. 3, 419–427 (July 1963*).
The principal result proved is that every normal congruence subgroup of the modular group of level prime to 6 is a principle congruence subgroup.

The meaning of "least" is least squares, C. Eisenhart, *J. Wash. Acad. Sci.* **54,** *No. 2, 24–33 (Feb. 1964*).
The Method of Least Squares was developed originally from three distinctly different points of view: (1) Least Sum of Squared Residuals (Legendre, 1805), (2) Maximum probability of Zero Error of Estimation (Gauss, 1809), and (3) Least Means Squared Error of Estimation (Gauss, 1821). These differ not only in their aims and in their initial assumptions, but also in the meanings that they attach to the numbers that all three yield as a common answer to any given problem. The evolution and implication of each of these three approaches to "Least Squares" are outlined.

Theory of irreversible processes in a plasma—derivation of a convergent kinetic equation from the generalized master equation, J. Weinstock, *Phys. Rev.* **133,** *No. 3A, A673–A680 (Feb. 1964*).
The theory of irreversible processes in plasmas is rigorously developed from the cluster formulation of the exact generalized Master equation. This equation is a time dependent analogue of the equilibrium virial expansion, and the development of the theory of non-equilibrium plasmas is found to parallel that of the well-known theory of equilibrium plasmas in a direct and simple way.
A *convergent* kinetic equation for homogeneous plasmas is then derived which includes the effects of close collisions as well as long-range collisions and is exact to first order in the density—in the asymptotic limit of long times. This kinetic equation is found to converge without ever introducing arbitrary cut-offs in the collision integrals or screened potentials—only coulomb potentials are dealt with. The divergence of previous kinetic equations is directly attributed to the neglect of significant close collision terms (such equations are not exact to first order in the density).
The method of derivation is quite general and the extension of the convergent kinetic equation to short times (non-Markoffian kinetic equation) as well as to general order in the density is indicated.

Statistical thermodynamics of the lattice model of a polymer molecule, J. Mazur, *J. Chem. Phys.* **40,** *No. 4, 1001–1007 (Feb. 15, 1964*).
Introduction of a potential energy of interaction between nonbonded neighbors of a lattice-simulated random-walk polymer enables us to study the thermodynamic properties of such a polymer. When the potential energy over a given range of intramolecular separations becomes infinitely large, the free energy and the entropy per unit step can be computed directly either from the attrition data using the Monte Carlo method of generating random walks, or from the exact numerical calculations for short chains. The transition to rod-like molecular configurations and the accompanying change in the free energy per unit step can be followed by gradually extending the range of the infinitely large intersegment interactions.
Instead of using computational methods for estimating the free energy of a polymer with a hard-core potential energy, one can approximate the free energy by limiting the intramolecular interactions to the nearest neighbors only. These interactions are limited to configurations separated by the smallest number of steps needed to bring two polymer segments within a specified range of a hard-core potential energy. The free energy can be calculated analytically, using certain recurrence relations constructed for the number of permissible configurations. A comparison is made of the results obtained on the basis of nearest neighbor approximation to the more accurate data obtained from the numerical calculations.

Antireciprocity and memory in the statistical approach to irreversible thermodynamics, R. E. Nettleton, *J. Chem. Phys.* **40,** *No. 1, 112–116 (Jan. 1964*).
The macroscopic state of a large, closed system is specified by the ensemble averages α_j of a set of even dynamical variables A_j, together with the averages ν_j of the quantities iLA_j, obtained by operating on the A_j with the self-adjoint Liouville operator L. Phenomenological equations for the time rates of change, $\dot{\alpha}_j$ and $\dot{\nu}_j$ are derived by a technique due to Zwanzig, in which one operates

on Liouville's equations with a projection operator which projects out the part which is "relevant" to the phenomenological description employed. These phenomenological equations are shown to exhibit the Onsager-Casimir reciprocity relations, including anti-symmetry relations whose derivation is found to require a modification of Zwanzig's mathematical assumptions. Since $\nu_j = \dot{\alpha}_j$, these equations also show that irreversible thermodynamics can be extended to the case where second-order time-derivatives appear, as well as non-linear terms in the α_j, provided the equations are still required to be linear in the ν_j. Furthermore, Onsager's equations are obtained under much less stringent restrictions on the variables and phenomenological matrices than appeared to be necessary in Zwanzig's original formulation.

Examples relating to the simplex method, A. J. Goldman and D. Kleinman, *Operations Res.* **12,** *No. 1, 159–161 (Jan.–Feb. 1964).*
An example is given showing that the most common variant of the simplex algorithm does not satisfy a condition conjectured by Saaty in an analysis of the maximum possible number of iterations. Situations are exhibited in which the simplex method encounters all vertices of the constraint polyhedron.

Precession equation of a spinning particle in nonuniform fields, U. Fano, *Phys. Rev.* **133,** *No. 3B, B828–B830 (Feb. 10, 1964).*
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Continuum theory of a plasma, J. L . Jackson and L. S. Klein, *Phys. Fluids* **7,** *No. 2, 232–241 (Feb. 1964).*
A continuum description of a classical plasma in equilibrium is presented. Thermodynamic properties and equilibrium fluctuations are discussed in terms of the fluctuations of the net charge density function, $n(r)$. The probability distribution function of the potential at a typical point, and the joint distribution of the potential at two points are calculated. As was noted in the preceding paper, these results lead to the chemical potential and two-particle distribution function of classical Debye-Hückel theory. A first order correction to the potential distribution of the continuum theory is obtained by treating the nearest neighbor charge as a discrete charge and the remainder of the plasma as a continuum. The corresponding correction to the free energy is obtained and compared with the results of diagrammatic calculations, to which the present results are similar.

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X-ray investigation of strain in cold-worked silver iodide. G. Burley.
Light scattering measurements on solutions of some quaternary ammonium salts. S. P. Wasik and W. D. Hubbard.
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Calorimetric properties of some alkali pentaborate hydrates from 15 to 370 °K. G. T. Furukawa, M. L. Reilly, and J. H. Piccirelli.
Rates of adsorption and desorption of polystyrene on chrome surface. R. R. Stromberg, W. H. Grant, and E. Passaglia.
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F-region irregularities studied by scintillation of signals from satellites. K. C. Yeh and G. W. Swenson, Jr.
Angels, insects, and weather. A. H. LaGrone, A. P. Deam, and G. B. Walker.
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